*Read*: Lax, Chapter 7, pages 89–100.

1. Let $A\colon X \to U$ be a linear map between finite-dimensional inner product spaces, and let $A^*\colon U \to X$ denote the adjoint. Prove each of the following equalities:
   (a) $N_{A^*} = R_A^\perp$
   (b) $R_{A^*} = N_A^\perp$
   (c) $N_A = R_{A^*}^\perp$
   (d) $R_A = N_{A^*}^\perp$.

2. Let $A\colon X \to U$ be a linear map between finite-dimensional inner product spaces. The map $A$ has a *left inverse* if there is a linear map $L\colon U \to X$ such that $LA = I_X$, the identity on $X$. It has a *right inverse* if there is a linear map $R\colon U \to X$ such that $AR = I_U$ is the identity on $U$.
   (a) Prove that $A$ maps $R_{A^*}$ bijectively onto $R_A$.
   (b) Show that if $A$ has a left inverse, then $Ax = u$ has *at most* one solution. Give a condition on $u$ that completely characterizes when there is a solution.
   (c) Show that if $A$ has a right inverse, then $Ax = u$ has *at least* one solution. If $Ax_p = u$ for some particular $x_p \in X$, then describe all solutions for $x$ in this case. What condition ensures that there will be *only* one solution?
   (d) What are the possibilities for the rank of $A$ if it has a left inverse? What if it has a right inverse?

3. Recall that a projection $P\colon X \to X$ is a linear map such that $P^2 = P$.
   (a) Show that if $P$ is a projection, then $X = R_P \oplus N_P$.
   (b) Show that $R_P^\perp = N_P$ if and only if $P$ is self-adjoint.

4. Consider four data points $(0, 0)$, $(1, 8)$, $(3, 8)$, and $(4, 20)$. In this problem, we will find the best fit line $C + Dt$ through these points, using least squares.
   (a) Write down an equation in matrix form, $Ax = b$, where $x = (C, D)^T$, that has no solution because these four points are not co-linear.
   (b) Find the best fit line by solving the related equation $A\hat{x} = p$, where $p$ is the orthogonal projection of $b$ onto the range of $A$.
   (c) Repeat the previous steps to find the best fit parabola $C + Dt + Et^2$.

5. Let $X$ be a finite-dimensional real inner product space. We say that a sequence $\{A_n\}$ of linear maps converges to a limit $A$ if $\lim_{n\to\infty} ||A_n - A|| = 0$.
   (a) Show that $\{A_n\}$ converges to $A$ if and only if for all $x \in X$, $A_n x$ converges to $Ax$.
   (b) Show by example that this fails if $\dim X = \infty$.