Read: Lax, Chapter 8, pages 101–120.

1. Let $N\colon X \to X$ be a normal mapping of an inner product space.
   (a) Prove that $||N|| = \max |n_i|$, where the n_is are the eigenvalues of N.
   (b) Show that N has a square-root, that is, a matrix R such that $N = R^2$. Is R necessarily normal? Unique?

2. Let $H\colon X \to X$ be self-adjoint, with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Prove the following *max-min* principle:
$$\lambda_k = \max_{\dim S = k} \min_{x \in S \setminus 0} R_H(x).$$

3. For any positive mapping $M\colon X \to X$, define an inner product on X by $\langle x, y\rangle := (x, My)$. Throughout this problem, assume that $X = \mathbb{R}^2$ and $M = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.
   (a) Find two orthonormal bases for X that contain the vector $e_1/||e_1||$, where $e_1 = (1, 0)$.
   (b) Find two orthonormal bases for X that contain the vector $e_2/||e_2||$, where $e_2 = (0, 1)$.
   (c) Find an vector v_2 orthogonal to $v_1 = (1, 1)$.
   (d) Find a matrix H that is self-adjoint with respect to $(\; , \;)$, but *not* with respect to $\langle \; , \; \rangle$.

4. Let $H, M\colon X \to X$ be self-adjoint mappings, and M positive.
   (a) Formulate and prove a necessary and sufficient condition for $M^{-1}H$ to be self-adjoint with respect to the standard inner product.
   (b) Prove that $M^{-1}H$ is self-adjoint with respect to the inner product $\langle x, y\rangle = (x, My)$. Conclude that there exists a basis $v_1, \dots, v_n$ of X and $\mu_1, \dots, \mu_n \in \mathbb{R}$ such that
$$Hv_i = \mu_i M v_i, \qquad \langle v_i, v_j\rangle = \delta_{ij}.$$
   (c) Find formulas for $\langle v_i, v_j\rangle$ and $\langle v_i, M^{-1}Hv_j\rangle$ in terms of the standard inner product.
   (d) Show that if H is positive, then so is $M^{-1}H$.

5. Let $H, M\colon X \to X$ be self-adjoint mappings, M positive, and define $R_{H,M}(x) = \dfrac{(x, Hx)}{(x, Mx)}$.
   (a) Show that $\mu_1 := \min\{R_{H,M}(x) \mid x \in X\}$ exists, and write an equation relating v_1, μ_1, H, and M.
   (b) Show that there is some $v_2 \in X$ solving the constrained minimum problem
$$\mu_2 := \min\big\{R_{H,M}(x) \mid (x, Mv_1) = 0\big\}.$$
   Write an equation relating v_2, μ_2, H, and M.
   (c) Find an invertible U and diagonal D such that $U^*MU = I$ and $U^*HU = D$.
   (d) Characterize the diagonal entries of D by a min-max principle.