Lecture 1.1: Vector spaces and linearity

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Algebraic structures

Definition

A group is a set G and associative binary operation $*$ with:

- **closure**: $a, b \in G$ implies $a * b \in G$;
- **identity**: there exists $e \in G$ such that $a * e = e * a = a$ for all $a \in G$;
- **inverses**: for all $a \in G$, there is $b \in G$ such that $a * b = e$.

A group is abelian if $a * b = b * a$ for all $a, b \in G$.

Definition

A field is a set $\mathbb{F}$ (or K) containing $1 \neq 0$ with two binary operations: $+$ (addition) and $\cdot$ (multiplication) such that:

(i) $\mathbb{F}$ is an abelian group under addition;

(ii) $\mathbb{F} \setminus \{0\}$ is an abelian group under multiplication;

(iii) The distributive law holds: $a(b + c) = ab + ac$ for all $a, b, c \in \mathbb{F}$.

Remarks

- $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$ (prime p), $\mathbb{Q}(\sqrt{2}) := \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ are all fields.
- $\mathbb{Z}$ is not a field. Nor is $\mathbb{Z}_n$ (composite n).
- the *additive identity* is 0, and the inverse of a is $-a$.
- the *multiplicative identity* is 1, and the inverse of a is a^{-1}, or $\frac{1}{a}$.

Vector spaces

Definition

A vector space is a set X ("vectors") over a field $\mathbb{F}$ ("scalars") such that:

(i) X is an abelian group under addition;

(ii) X is closed under scalar multiplication;

(iii) $+$ and $\cdot$ are "compatible" via natural associative and distributive laws relating the two:

- $a(bv) = (ab)v$, for all $a, b \in \mathbb{F}$, $v \in X$;
- $a(v + w) = av + aw$, for all $a \in \mathbb{F}$, $v, w \in X$;
- $(a + b)v = av + bv$, for all $a, b \in \mathbb{F}$, $v \in X$;
- $1v = v$, for all $v \in X$.

Intuition

Think of a vector space as a set of vectors that is:

(i) Closed under addition, subtraction, and scalar multiplication;

(ii) Equipped with the "natural" associative and distributive laws.

Proposition (exercise)

In any vector space X,

(i) The zero vector $\mathbf{0}$ is unique;

(ii) $0x = \mathbf{0}$ for all $x \in X$;

(iii) $(-1)x = -x$ for all $x \in X$. □

Linear maps

Definition

A linear map between vector spaces X and Y over $\mathbb{F}$ is a function $\varphi\colon X \to Y$ satisfying:

- $\varphi(v + w) = \varphi(v) + \varphi(w)$, for all $v, w \in X$;
- $\varphi(av) = a\,\varphi(v)$, for all $a \in \mathbb{F}$, $v \in X$.

An isomorphism is a linear map that is bijective (1–1 and onto).

Proposition

The two conditions for linearity above can be replaced by the single condition:

$$\varphi(av + bw) = a\varphi(v) + b\varphi(w), \qquad \text{for all } v, w \in X \text{ and } a, b \in \mathbb{F}.$$

Examples of vector spaces

(i) $K^n = \{(a_1, \dots, a_n) : a_i \in K\}$. Addition and multiplication are defined componentwise.

(ii) Set of functions $\mathbb{R} \longrightarrow \mathbb{R}$ (with $K = \mathbb{R}$).

(iii) Set of functions $S \longrightarrow K$ for an abitrary set S.

(iv) Set of polynomials of degree $< n$, with coefficients from K.

Exercise

In the list of vector spaces above, (i) is isomorphic to (iv), and to (iii) if $|S| = n$. $\square$

Subspaces

Definition

A subset Y of a vector space X is a subspace if it too is a vector space.

Examples

(i) $Y = \{(0, a_2, \dots, a_{n-1}, 0) : a_i \in K\} \subseteq K^n$.

(ii) $Y = \{\text{functions with period } T \big| \pi\} \subseteq \{\text{functions } \mathbb{R} \to \mathbb{R}\}$.

(iii) $Y = \{\text{constant functions } S \to K\} \subseteq \{\text{functions } S \to K\}$.

(iv) $Y = \{a_0 + a_2x^2 + a_4x^4 + \cdots + a_{n-1}x^{n-1} : a_i \in K\} \subseteq \{\text{polynomials of degree} < n\}$.

Definition

If Y and Z are subsets of a vector space X, then their:

- sum is $Y + Z = \{y + z \mid y \in Y,\ z \in Z\}$;
- intersection is $Y \cap Z = \{x \mid x \in Y,\ x \in Z\}$.

Exercise

If Y and Z are subspaces of X, then $Y + Z$ and $Y \cap Z$ are also subspaces. □