Lecture 2.3: Algebra of linear mappings

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Basic definitions and properties

Definition

Let $S, T\colon X \to U$ be linear maps. Define

- $T + S$ by $(T+S)(x) = Tx + Sx$ for each $x \in X$.
- aT by $(aT)(x) = T(ax)$ for each $x \in X$, $a \in K$.

Easy fact

The set of linear maps from $X \to U$, denoted $\mathrm{Hom}(X,U)$, or $\mathscr{L}(X,U)$, is a vector space.

Lemma 2.3 (HW)

If $T\colon X \to U$ and $S\colon U \to V$ are linear maps, then so is $(S \circ T)\colon X \to V$.

Moreover, composition is distributive w.r.t. addition. That is, if $P, T\colon X \to U$ and $R, S\colon U \to V$, then

$$(R+S)\circ T = R\circ T + S\circ T, \qquad S\circ(T+P) = S\circ T + S\circ P\,.$$

Remarks

- We usually just write $S \circ T$ as just ST.
- In general, $ST \neq TS$ (note that TS may not even be defined).

Invertibility

Definition

A linear map T is invertible if it is 1–1 and onto (i.e., if it is an isomorphism). Denote the inverse by T^{-1}.

Exercise

If T is invertible, then TT^{-1} is the identity.

Proposition 2.4 (exercise)

Let $T \colon X \to U$ be linear.

(i) If T is linear, then so is T^{-1}.

(ii) If S and T are invertible and ST defined, then it is invertible with $(ST)^{-1} = T^{-1}S^{-1}$.

Examples

(ix) Take $X = U = V = \mathbb{R}[t]$, with $T = \frac{d}{dt}$ and $S =$ multiplication by t.

(x) Take $X = U = V = \mathbb{R}^3$, with S a 90°-rotation around the x_1 axis, and T a 90°-rotation around the x_2 axis.

In both of these examples, S and T are linear with $ST \neq TS$. (Which are invertible?)

Some more advanced concepts

Definition

An endomorphism of X is a linear map from X to itself. Denote the set of endomorphisms of X by $\text{Hom}(X, X)$ or $\mathscr{L}(X, X)$ or $\text{End}(X)$.

Remarks

$\text{Hom}(X, X)$ is a vector space, but we can also "multiply" vectors; it is an algebra.

It is an associative but noncommutative algebra, with unity I, satisfying $Ix = x$.

$\text{Hom}(X, X)$ contains zero divisors: pairs S, T such that $ST = 0$ but neither S nor T is zero.

Proposition

If $A \in \text{Hom}(X, X)$ is a left inverse of $B \in \text{Hom}(X, X)$ [i.e., $AB = I$], then it is also a right inverse [i.e., $BA = I$]. □

Definition

The invertible elements of $\text{Hom}(X, X)$ forms the general linear group, denoted $\text{GL}_n(K)$, where $n = \dim X$.

Every $S \in \text{GL}_n(K)$ defines a similarity transformation of $\text{Hom}(X, X)$, sending $M \longmapsto M_S := SMS^{-1}$, for each $M \in \text{Hom}(X, X)$. We say M and M_S are similar.

Similarity

Proposition 2.5

Every similarity transform is an automorphism ["structure-preserving bijection"] of $\text{Hom}(X, X)$:

$$(kM)_S = kM_S, \qquad (M+N)_S = M_S + N_S, \qquad (MN)_S = M_S N_S\,.$$

Moreover, the set of similarity transforms forms a group under $(M_S)_T := M_{TS}$, called the inner automorphism group of $GL_n(K)$.

Proof

Proposition 2.6 (exercise)

Similarity is an equivalence relation, i.e., it is:

(i) Reflexive: $M \sim M$;

(ii) Symmetric: $L \sim M$ implies $M \sim L$;

(iii) Transitive: $L \sim M$ and $M \sim N$ implies $L \sim N$. □

Last but not least

Proposition 2.7 (HW)

If either A or B in $\mathrm{Hom}(X,X)$ is invertible, then AB and BA are similar. □

Given any $A \in \mathrm{Hom}(X,X)$ and polynomial $p(s) = a_N s^N + \cdots + a_1 s + a_0$, consider the polynomial $p(A) = a_N A^N + \cdots + a_1 A + a_0 I$.

The set of polynomials in A is a commutative subalgebra of $\mathrm{Hom}(X,X)$. [to be revisited]

Miscellaneous definitions

- A linear map $P \colon X \to X$ is a projection if $P^2 = P$.
- The commutator of $A, B \in \mathrm{Hom}(X,X)$ is $[A,B] := AB - BA$, which is 0 iff A and B commute.

Examples (contin.)

(xii) If $X = \{f : \mathbb{R} \to \mathbb{R},\ \text{contin.}\}$, then the following maps $P, Q \in \mathrm{Hom}(X,X)$ are projections:

- $(Pf)(x) = \dfrac{f(x) + f(-x)}{2}$; this is the even part of f.
- $(Qf)(x) = \dfrac{f(x) - f(-x)}{2}$; this is the odd part of f.

Note that $f = Pf + Qf$ for any $f \in X$.