Lecture 2.4: The four subspaces

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Math 8530, Advanced Linear Algebra

Overview

Over several upcoming lectures, we will learn about the transpose of a linear map, and then how to encode an arbitrary linear map with a matrix.

Though it is not necessary, it is helpful to have some familiarity with undergraduate-level matrix analysis *before* seeing these.

In this lecture, we will review the "four subspaces" that arise from every matrix:

1. column space
2. row space
3. nullspace
4. left nullspace

Understanding these subspaces will motivate the more theoretical concepts and results as they arise, and give them context.

Throughout, we strongly recommend viewing:

1. vectors $x \in X$ as column vectors
2. scalar functions $\ell \in X'$ as row vectors.

Column space, and row space

Let $A\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, which we can think of as an $m \times n$ matrix, $A = (a_{ij})$.

The transpose is a linear map $A^T : \mathbb{R}^m \to \mathbb{R}^n$, which we can think of as an $n \times m$ matrix.

Definition

The range R_A is the span of the column vectors, called the column space of A.

Its dimension is called the column rank of A.

The range R_{A^T} is the span of the column vectors of A^T, called the row space of A.

Its dimension is called the row rank of A.

Theorem

$\dim R_A = \dim R_{A^T}$, which we call the rank of A.

Moreover, the restriction of $A\colon R_{A^T} \to R_A$ is bijective.

Column space, row space, nullspace and left nullspace

Four ways to multiply matrices

Suppose we have linear maps $\mathbb{R}^p \xrightarrow{B} \mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$. As matrices, we can multiply them:

1. rows by columns
2. by columns
3. by rows
4. columns by rows

Systems of equations and Gaussian elimination

Let's review how to solve a system of equations, and how it relates to the 4 subspaces.

$$\begin{aligned} x_1 + x_2 + 2x_3 + 3x_4 &= u_1 \\ x_1 + 2x_2 + 3x_3 + x_4 &= u_2 \\ 2x_1 + x_2 + 2x_3 + 3x_4 &= u_3 \\ 3x_1 + 4x_2 + 6x_3 + 2x_4 &= u_4 \end{aligned}$$