Lecture 3.1: Determinant prerequisites

Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 8530, Advanced Linear Algebra

What is a determinant

Definition (unofficial)

The determinant of $T \colon \mathbb{R}^n \to \mathbb{R}^n$ is the signed volume of $T([0,1]^n)$, the image of the unit n-cube.

Permutations

Definition

Let $[n] := \{1, \ldots, n\}$. A permutation is a bijection $\pi \colon [n] \to [n]$. The set of all $n!$ permutations is the symmetric group, S_n.

Definition

The discriminant of variables $x_1, \ldots, x_n$ is

$$P(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j).$$

Permuting variables only changes the sign of the discriminant:

$$P\big(\pi(x_1, \ldots, x_n)\big) = \prod_{i<j} (x_{\pi(i)} - x_{\pi(j)}) = \underbrace{\mathrm{sgn}(\pi)}_{\pm 1} \prod_{i<j} (x_i - x_j).$$

We call $\mathrm{sgn}(\pi)$ the sign of the permutation π.

Transpositions

A transposition is a permutation $\tau \in S_n$ that swaps two entries and fixes the rest. That is,

$$\tau(i) = j, \quad \tau(j) = i, \qquad \tau(k) = k, \text{ if } k \neq i, j.$$

We write this as (ij).

Proposition (HW)

(i) $\operatorname{sgn}(\pi_1 \circ \pi_2) = \operatorname{sgn}(\pi_1)\operatorname{sgn}(\pi_2)$

(ii) $\operatorname{sgn}(\tau) = -1$ for any transposition

(iii) every $\pi \in S_n$ can be written as a composition of transpositions: $\pi = \tau_k \circ \cdots \circ \tau_1$

(iv) the parity of this decomposition is unique

(v) if $\pi = \tau_k \circ \cdots \circ \tau_1$, then $\operatorname{sgn}(\pi) = (-1)^k$.