Lecture 4.3: Generalized eigenvectors

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Math 8530, Advanced Linear Algebra

Definitions

Throughout, $A\colon X \to X$ will be an $n \times n$ matrix over an algebraically closed field K.

Let I be the set of polynomials

$$I = \{p(t) \in \mathbb{C}[t] \mid p(A) = 0\}.$$

This is an ideal of $\mathbb{C}[t]$ since it's closed under addition, subtraction, and scalar multiplication.

Since $\mathbb{C}[t]$ is a principal ideal domain (PID), I is generated by a single element.

That is, $I = \langle m_A(t) \rangle$, for some monic polynomial $m_A(t)$, called the minimal polynomial of A.

All polynomials $p(t)$ such that $p(A) = 0$ are multiples of $m_A(t)$.

Let's verify existence and uniqueness of $m_A(t)$ without using ring theoretic ideas.

2×2 examples

Examples

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
2. $A = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$

Remark

Every 2×2 matrix with $\operatorname{tr} A = 2$ and $\det A = 1$ has $\lambda = 1$ as a double root of $p_A(t)$. These matrices form a 2-parameter family of $p_A(t)$, and only $A = I$ has two linearly independent eigenvectors.

3×3 examples

Suppose A is a 3×3 matrix and $p_A(t) = (t-1)^3$. Since $m_A(t)$ divides $p_A(t)$, there are three possibilities:

1. $m_A(t) = t - 1$
2. $m_A(t) = (t-1)^2$
3. $m_A(t) = (t-1)^3$.

Generalized eigenvectors

Suppose λ is an eigenvalue with multiplicity m, but only one eigenvector $v \in X$. Then

$$(A-\lambda I)v_1 = 0, \qquad \dim N_{A-\lambda I} = 1, \qquad \operatorname{rank}(A-\lambda I) = m-1.$$

Big idea

We can *always* find some $v_2 \in X$ such that

$$(A-\lambda I)v_2 = v_1, \qquad \Longrightarrow \qquad (A-\lambda I)^2 v_2 = 0.$$

Similarly, we can find $v_3 \in X$ such that

$$(A-\lambda I)v_3 = v_2, \qquad \Longrightarrow \qquad (A-\lambda I)^3 v_3 = 0, \quad \text{but} \quad (A-\lambda I)^2 v_3 = v_1 \neq 0.$$

Definition

A vector $v \in X$ is a generalized eigenvector of A with eigenvalue λ if $(A-\lambda I)^m v = 0$ for some $m \geq 1$. The "genuine" eigenvectors are when $m = 1$.

2×2 examples, revisited

Examples

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2. $A = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$