Lecture 5.5: Projection and least squares

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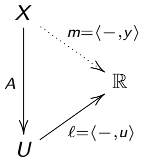
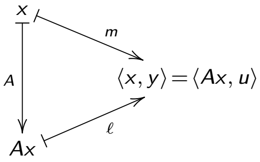
Self-adjointness

Recall that the adjoint of A is the map $A^*\colon U \to X$ such that

$$\underbrace{\langle x, A^* u\rangle}_{\text{inner product in } X} = \underbrace{\langle Ax, u\rangle}_{\text{inner product in } U}.$$





Definition

A linear map $A\colon X \to U$ is self-adjoint if $A^* = A$.

Proposition 5.9

The linear maps A^*A and AA^* are self-adjoint.

Projections and orthogonal

Recall that if $X = Y \oplus Y^\perp$, then the map

$$P_Y\colon X \longrightarrow X, \qquad P_Y\colon y + y^\perp \longmapsto y$$

is the orthogonal projection of X onto Y.

Proposition 5.10

Orthogonal projections are self-adjoint.

Some books define a projection to be any linear map $P\colon X \to X$ such that $P^2 = P$.

It is not hard to show that $X = R_P \oplus N_P$.

Exercise (HW)

A projection $P\colon X \to X$ is an orthogonal projection if and only if it is self-adjoint.

More on the map A^*A

Lemma 5.11

The maps A and A^*A have the same nullspace.

Suppose A is an $m \times n$ matrix ($m > n$) with linearly independent columns. Then:

- the columns of A are a *basis* for the range (column space) of A
- A^*A is invertible.

The map A^*A and projection

The fact that $N_{A^*A} = N_A$, and the following, is the crux of the least squares method of finding the "best fit line."

Corollary 5.12

Consider an underdetermined system $Ax = b$, where $A \colon X \to U$ has trivial nullspace. The (unique) vector x that minimizes $||Ax - b||^2$ is the solution to $A^*Az = A^*b$.

An example of least squares

Let's find the "best fit line" $a_0 + a_1 x$ through the points $(1,1)$, $(2,2)$, and $(3,2)$ in $\mathbb{R}^2$.

The projection map $A(A^*A)^{-1}A^*$

Key idea

Let $y_1, \dots, y_k$ be a basis for Y, and $A = [y_1\ y_2\ \cdots\ y_k]$. Then

$$A(A^*A)^{-1}A^*$$

is the orthogonal projection matrix onto Y.