Lecture 6.5: Self-adjoint differential operators

Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 8530, Advanced Linear Algebra

Overview

In an earlier lecture, we gave examples of orthogonal functions arising from differential equations (ODEs).

The reason *why* they exist is because they are eigenfunctions of a self-adjoint differential operator.

This is the idea of Sturm-Liouville theory, which we will summarize here.

We will not assume any knowledge about differential equations, other than what they are.

For more detailed information, see my series of lectures on *Advanced Engineering Mathematics*.

Self-adjointness of the SL operator

Definition

A Sturm-Liouville equation is a 2nd order ODE of the following form:

$$-\frac{d}{dx}\Big(p(x)y'\Big) + q(x)y = \lambda w(x)y, \qquad \text{where } p(x),\ q(x),\ w(x) > 0.$$

We are usually interested in solutions $y(x)$ on $[a, b]$, under homogeneous BCs:

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \qquad \alpha_1^2 + \alpha_2^2 > 0$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0 \qquad \beta_1^2 + \beta_2^2 > 0.$$

Together, this BVP is called a Sturm-Liouville (SL) problem.

Remark

Consider the linear differential operator $L = \dfrac{1}{w(x)}\Big(-\dfrac{d}{dx}\Big[p(x)\dfrac{d}{dx}\Big] + q(x)\Big)$.

$$\mathbb{C}^\infty[a,b] \xrightarrow{L_1 = p(x)\frac{d}{dx}} \mathbb{C}^\infty[a,b] \xrightarrow{L_2 = -\frac{1}{w(x)}\frac{d}{dx} + \frac{q(x)}{w(x)}} \mathbb{C}^\infty[a,b]$$

$$y \longmapsto p(x)y'(x) \longmapsto \frac{-1}{w(x)}\frac{d}{dx}\big[p(x)y'(x)\big] + \frac{q(x)}{w(x)}y(x)$$

An SL equation is just an eigenvalue equation: $Ly = \lambda y$, and $L = L_2 \circ L_1$ is self-adjoint!

Main theorem

The SL operator $L = \frac{1}{w(x)}\Big(-\frac{d}{dx}\Big[p(x)\frac{d}{dx}\Big] + q(x)\Big)$ is self-adjoint on $\mathcal{C}^\infty_{\alpha,\beta}[a,b]$ with respect to the inner product

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}w(x)\,dx.$$

This means that:

(a) The eigenvalues are real and can be ordered so $\lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty$.

(b) Each eigenvalue λ_i has a unique (up to scalars) eigenfunction $y_i(x)$.

(c) W.r.t. the inner product $\langle f, g\rangle := \int_a^b f(x)\overline{g(x)}w(x)\,dx$, the eigenfunctions form an orthogonal basis on the subspace of functions $\mathcal{C}^\infty_{\alpha,\beta}[a,b]$ that satisfy the BCs.

Definition

If $f \in \mathcal{C}^\infty_{\alpha,\beta}[a,b]$, then f can be written uniquely as a linear combination of the eigenfunctions. That is,

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \qquad \text{where } c_n = \frac{\langle f, y_n\rangle}{\langle y_n, y_n\rangle} = \frac{\int_a^b f(x)\overline{y_n(x)}w(x)\,dx}{\int_a^b ||y_n(x)||^2 w(x)\,dx}.$$

This is called a generalized Fourier series with respect to the orthogonal basis $\{y_n(x)\}$ and weighting function $w(x)$.

Fourier series

Dirichlet BCs

$-y'' = \lambda y, \quad y(0) = 0, \quad y(\pi) = 0$ is an SL problem with:

- Eigenvalues: $\lambda_n = n^2, \quad n = 1, 2, 3, \dots$.
- Eigenfunctions: $y_n(x) = \sin(nx)$.

The orthogonality of the eigenvectors means that

$$\langle y_m, y_n \rangle := \int_0^\pi y_m(x) y_n(x) w(x)\, dx = \int_0^\pi \sin(mx)\, \sin(nx)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi/2 & \text{if } m = n. \end{cases}$$

Note that this means that $||y_n|| := \langle y_n, y_n \rangle^{1/2} = \sqrt{\pi/2}$.

Fourier series: *any* function $f(x)$, continuous on $[0, \pi]$ satisfying $f(0) = 0, \; f(\pi) = 0$ can be written *uniquely* as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{\langle f, \sin nx \rangle}{\langle \sin nx, \sin nx \rangle} = \frac{\int_0^\pi f(x) \sin nx\, dx}{|| \sin nx ||^2} = \frac{2}{\pi} \int_0^\pi f(x) \sin nx\, dx.$$

Fourier series

Neumann BCs

$-y'' = \lambda y, \quad y'(0) = 0, \quad y'(\pi) = 0$ is an SL problem with:

- Eigenvalues: $\lambda_n = n^2, \quad n = 0, 1, 2, 3, \ldots$.
- Eigenfunctions: $y_n(x) = \cos(nx)$.

The orthogonality of the eigenvectors means that

$$\langle y_m,\, y_n\rangle := \int_0^\pi y_m(x) y_n(x) w(x)\, dx = \int_0^\pi \cos(mx)\, \cos(nx)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi/2 & \text{if } m = n > 0. \end{cases}$$

Note that this means that $||y_n|| := \langle y_n,\, y_n\rangle^{1/2} = \begin{cases} \sqrt{\pi/2} & n > 0 \\ \sqrt{\pi} & n = 0. \end{cases}$

Fourier series: *any* function $f(x)$, continuous on $[0, \pi]$ satisfying $f'(0) = 0, \;\; f'(\pi) = 0$ can be written *uniquely* as

$$f(x) = \sum_{n=0}^{\infty} a_n \cos nx$$

where

$$a_n = \frac{\langle f,\, \cos nx\rangle}{\langle \cos nx,\, \cos nx\rangle} = \frac{\int_0^\pi f(x)\cos nx\, dx}{||\cos nx||^2} = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\, dx.$$

More complicated Sturm-Liouville problems

Every 2nd order linear homogeneous ODE, $y'' + P(x)y' + Q(x)y = 0$ can be written in self-adjoint or "Sturm-Liouville form":

$$-\frac{d}{dx}\Big(p(x)y'\Big) + q(x)y = \lambda w(x)y, \qquad \text{where } p(x),\ q(x),\ w(x) > 0.$$

Examples from physics and engineering

- Legendre's equation: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$. *Used for modeling spherically symmetric potentials in the theory of Newtonian gravitation and in electricity & magnetism (e.g., the wave equation for an electron in a hydrogen atom).*
- Parametric Bessel's equation: $x^2y'' + xy' + (\lambda x^2 - \nu^2)y = 0$. *Used for analyzing vibrations of a circular drum.*
- Chebyshev's equation: $(1-x^2)y'' - xy' + n^2y = 0$. *Arises in numerical analysis techniques.*
- Hermite's equation: $y'' - 2xy' + 2ny = 0$. *Used for modeling simple harmonic oscillators in quantum mechanics.*
- Laguerre's equation: $xy'' + (1-x)y' + ny = 0$. *Arises in a number of equations from quantum mechanics.*
- Airy's equation: $y'' - k^2xy = 0$. *Models the refraction of light.*

Legendre's differential equation

Consider the following Sturm-Liouville problem, defined on $(-1, 1)$:

$$-\frac{d}{dx}\Big[(1-x^2)\frac{d}{dx}y\Big] = \lambda y, \qquad \Big[p(x) = 1-x^2, \quad q(x) = 0, \quad w(x) = 1\Big].$$

The eigenvalues are $\lambda_n = n(n+1)$, $n \in \mathbb{N}$, and the eigenfunctions solve Legendre's equation:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

For each n, one solution is a degree-n "Legendre polynomial"

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\big[(x^2-1)^n\big].$$

They are orthogonal with respect to the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

It can be checked that

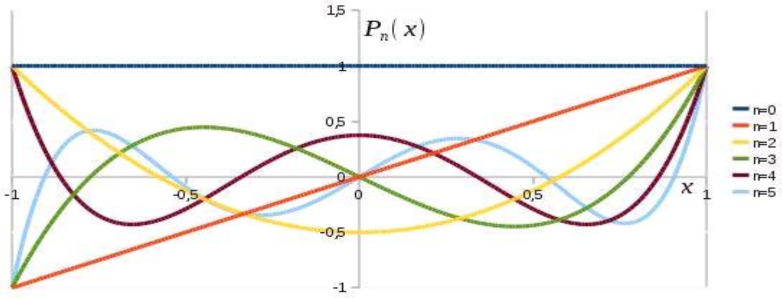
$$\langle P_m, P_n\rangle = \int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1}\delta_{mn}.$$

By orthogonality, every function f, continuous on $-1 < x < 1$, can be expressed using Legendre polynomials:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \qquad \text{where} \quad c_n = \frac{\langle f,\, P_n\rangle}{\langle P_n,\, P_n\rangle} = (n+\tfrac{1}{2})\,\langle f,\, P_n\rangle$$

Legendre polynomials

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \\
P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \\
P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) &= \tfrac{1}{8}(63x^5 - 70x^3 + 15x) \\
P_6(x) &= \tfrac{1}{8}(231x^6 - 315x^4 + 105x^2 - 5) \\
P_7(x) &= \tfrac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)
\end{aligned}$$



Parametric Bessel's differential equation

Consider the following Sturm-Liouville problem on $[0, a]$:

$$-\frac{d}{dx}(xy') - \frac{\nu^2}{x}y = \lambda xy, \qquad \Big[p(x) = x, \quad q(x) = -\frac{\nu^2}{x}, \quad w(x) = x\Big].$$

For a fixed ν, the eigenvalues are $\lambda_n = \omega_n^2 := \alpha_n^2/a^2$, for $n = 1, 2, \dots$.

Here, α_n is the n^{th} positive root of $J_\nu(x)$, the Bessel functions of the first kind of order ν.

The eigenfunctions solve the parametric Bessel's equation:

$$x^2y'' + xy' + (\lambda x^2 - \nu^2)y = 0.$$

Fixing ν, for each n there is a solution $J_{\nu n}(x) := J_\nu(\omega_n x)$.

They are orthogonal with repect to the inner product $\langle f, g\rangle = \int_0^a f(x)g(x)\,x\,dx$.

It can be checked that

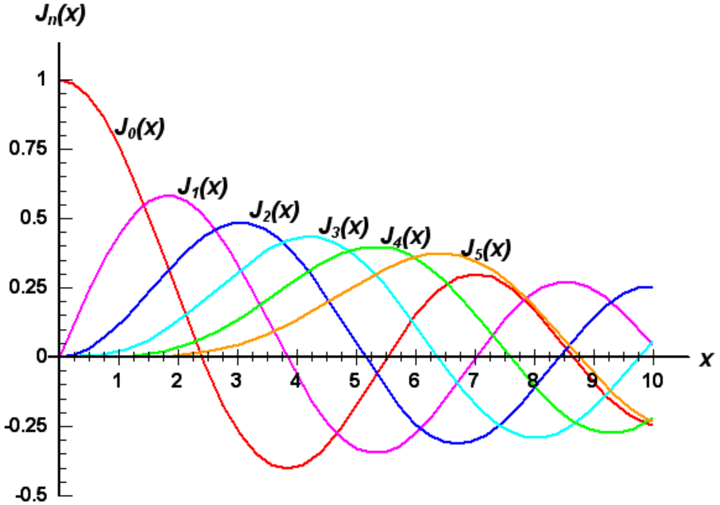
$$\langle J_{\nu n}, J_{\nu m}\rangle = \int_0^a J_\nu(\omega_n x)J_\nu(\omega_m x)\,x\,dx = 0, \quad \text{if } n \neq m.$$

By orthogonality, every continuous function $f(x)$ on $[0, a]$ can be expressed in a "Fourier-Bessel" series:

$$f(x) \sim \sum_{n=0}^{\infty} c_n J_\nu(\omega_n x), \qquad \text{where} \quad c_n = \frac{\langle f,\ J_{\nu n}\rangle}{\langle J_{\nu n},\ J_{\nu n}\rangle}.$$

Bessel functions (of the first kind)

$$J_\nu(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!(\nu+m)!} \left(\frac{x}{2}\right)^{2m+\nu} .$$



Chebyshev's differential equation

Consider the following Sturm-Liouville problem on $[-1,1]$:

$$-\frac{d}{dx}\left[\sqrt{1-x^2}\,\frac{d}{dx}y\right] = \lambda\frac{1}{\sqrt{1-x^2}}y, \qquad \left[p(x)=\sqrt{1-x^2}, \quad q(x)=0, \quad w(x)=\tfrac{1}{\sqrt{1-x^2}}\right].$$

The eigenvalues are $\lambda_n = n^2$ for $n \in \mathbb{N}$, and the eigenfunctions solve Chebyshev's equation:

$$(1-x^2)y'' - xy' + n^2 y = 0.$$

For each n, one solution is a degree-n "Chebyshev polynomial," defined recursively by

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

They are orthogonal with repect to the inner product $\langle f, g\rangle = \displaystyle\int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\,dx$.

It can be checked that

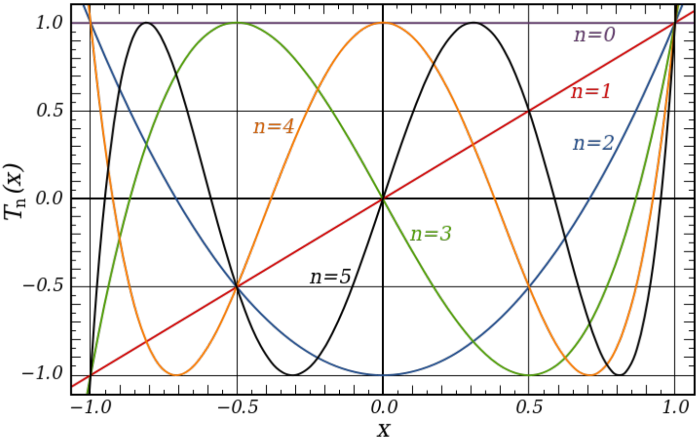
$$\langle T_m, T_n\rangle = \int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} \frac{1}{2}\pi\delta_{mn} & m\neq 0, n\neq 0 \\ \pi & m=n=0 \end{cases}$$

By orthogonality, every function $f(x)$, continuous for $-1<x<1$, can be expressed using Chebyshev polynomials:

$$f(x) \sim \sum_{n=0}^{\infty} c_n T_n(x), \qquad \text{where} \quad c_n = \frac{\langle f,\ T_n\rangle}{\langle T_n,\ T_n\rangle} = \frac{2}{\pi}\langle f,\ T_n\rangle, \quad \text{if } n>0.$$

Chebyshev polynomials (of the first kind)

$$T_0(x) = 1 \qquad T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_1(x) = x \qquad T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_2(x) = 2x^2 - 1 \qquad T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$
$$T_3(x) = 4x^3 - 3x \qquad T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$



Hermite's differential equation

Consider the following Sturm-Liouville problem on $(-\infty, \infty)$:

$$-\frac{d}{dx}\left[e^{-x^2}\frac{d}{dx}y\right] = \lambda e^{-x^2}y, \qquad \left[p(x) = e^{-x^2}, \quad q(x) = 0, \quad w(x) = e^{-x^2}\right].$$

The eigenvalues are $\lambda_n = 2n$ for $n = 1, 2, \dots,$ and the eigenfunctions solve Hermite's equation:

$$y'' - 2xy' + 2ny = 0.$$

For each n, one solution is a degree-n "Hermite polynomial," defined by

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} = \left(2x - \frac{d}{dx}\right)^n \cdot 1$$

They are orthogonal with repect to the inner product $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,dx.$

It can be checked that

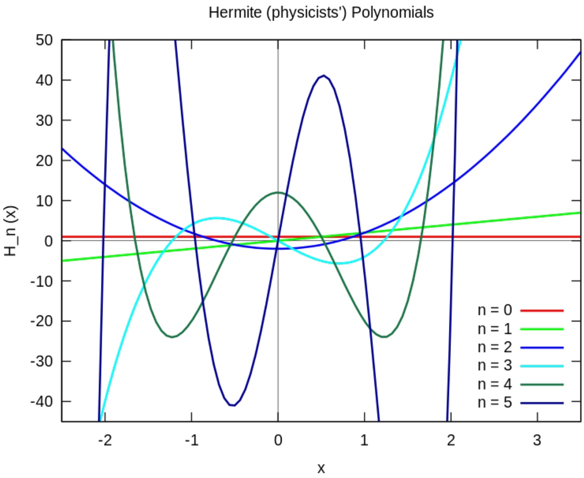
$$\langle H_m, H_n\rangle = \int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx = \sqrt{\pi}2^n n!\delta_{mn}.$$

By orthogonality, every function $f(x)$ satisfying $\int_{-\infty}^{\infty} f^2e^{-x^2}dx < \infty$ can be expressed using Hermite polynomials:

$$f(x) \sim \sum_{n=0}^{\infty} c_nH_n(x), \qquad \text{where} \quad c_n = \frac{\langle f,\ H_n\rangle}{\langle H_n,\ H_n\rangle} = \frac{\langle f, H_n\rangle}{\sqrt{\pi}2^n n!}.$$

Hermite polynomials

$$H_0(x) = 1 \qquad H_4(x) = 16x^4 - 48x^2 + 12$$
$$H_1(x) = 2x \qquad H_5(x) = 32x^5 - 160x^3 + 120x$$
$$H_2(x) = 4x^2 - 2 \qquad H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$$
$$H_3(x) = 8x^3 - 12x \qquad H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$$



Hermite functions

The Hermite functions can be defined from the Hermite polynomials as

$$\psi_n(x) = \left(2^n n!\sqrt{\pi}\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x) = (-1)^n \left(2^n n!\sqrt{\pi}\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-x^2}.$$

They are orthonormal with respect to the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\, dx.$$

Every real-valued function f such that $\int_{-\infty}^{\infty} f^2\, dx < \infty$ "can be expressed uniquely" as

$$f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x)\, dx, \qquad \text{where } c_n = \langle f, \psi_n\rangle = \int_{-\infty}^{\infty} f(x)\psi_n(x)\, dx.$$

These are solutions to the time-independent Schrödinger ODE: $-y'' + x^2 y = (2n+1)y$.