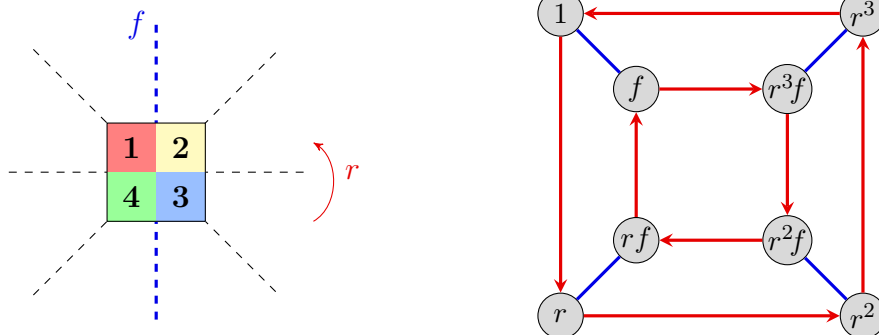
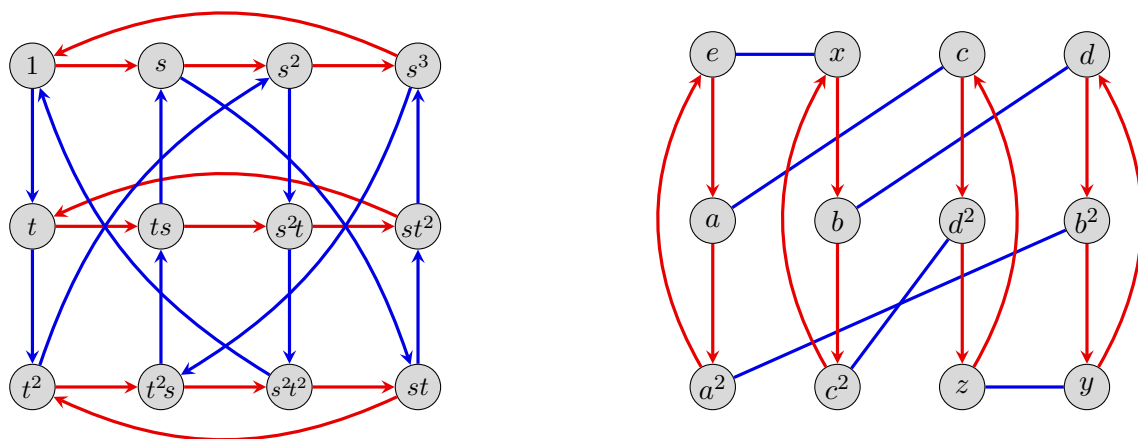


1. The eight symmetries of a square form a group that we will call \mathbf{Sq} , generated by a 90° counterclockwise rotation r , and a horizontal flip f . A Cayley diagram is shown below.



- For each axis of reflection, express the symmetry across it in terms of r and f .
 - Find all *minimal* generating sets. [Hint: There are 12.]
 - Let $s = f$ and $t = r^3f = fr$. Draw a Cayley diagram using s and t as generators.
 - Write a presentation of the form $\mathbf{Sq} = \langle r, f \mid \dots \rangle$.
 - Write a presentation of the form $\mathbf{Sq} = \langle s, t \mid \dots \rangle$.
 - Construct a *Cayley table* for this group, ordered $1, r, r^2, r^3, f, rf, r^2f, r^3f$. Describe how the rotations and reflections are “clustered” in this table.
2. The Cayley diagrams of two groups of size 12 are shown below.



- Create a Cayley table for each group. (For consistency, please order the elements in the first group by $1, t^2, s^2t, t, s^2, s^2t^2, s, st^2, t^2s, st, s^3, ts$, and those in second by $e, x, y, z, a, b, c, d, a^2, b^2, c^2, d^2$.)
- Find the inverse of each element.
- Find the *order* of each g : the minimal $k > 0$ such that $g^k = e$, denoted $|g|$.
- Write a presentation for each group.
- Determine whether or not these two groups are isomorphic. Justify your answer.
- Squint your eyes. Do you see any patterns in these tables?

3. In this problem, we will define two variations of the **Coin**₂ group from lecture. We will consider two types of tiles, and declare the following to be the “home state” of each:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array}$$

Our first group is **Coin**₃ = $\langle c, t \rangle$, where c “cyclically shifts” the entries, and t “toggles” the color of the leftmost square:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \xrightarrow{c} \begin{array}{|c|c|c|} \hline 3 & 1 & 2 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \xrightarrow{t} \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}$$

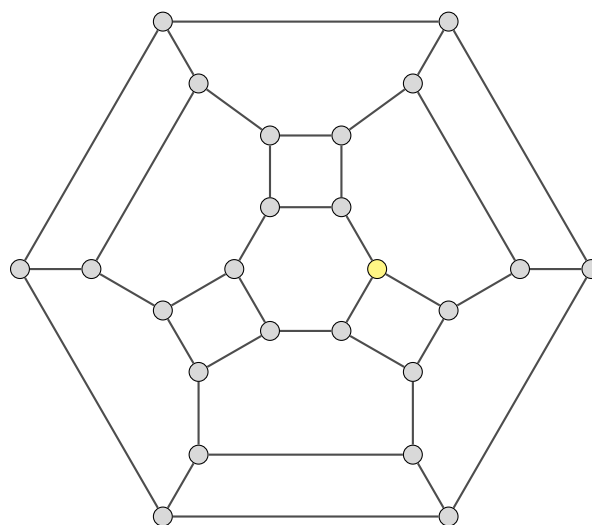
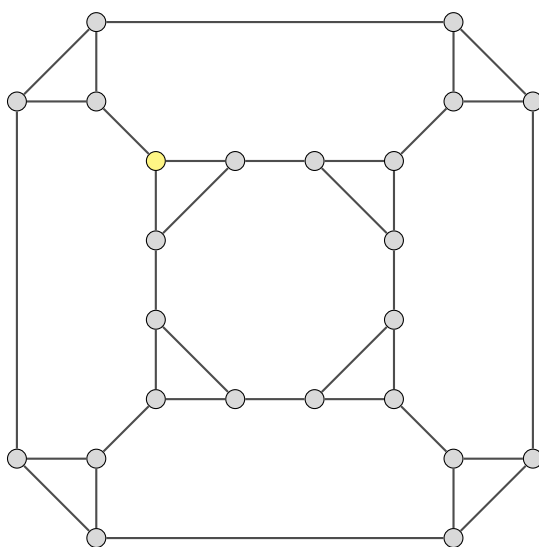
Our second group is **Box**₂ = $\langle r, s \rangle$, where r “rotates” the squares counterclockwise, and s “swaps” the squares on the top row.

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} \xrightarrow{r} \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array} \xrightarrow{s} \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 4 & 3 \\ \hline \end{array}$$

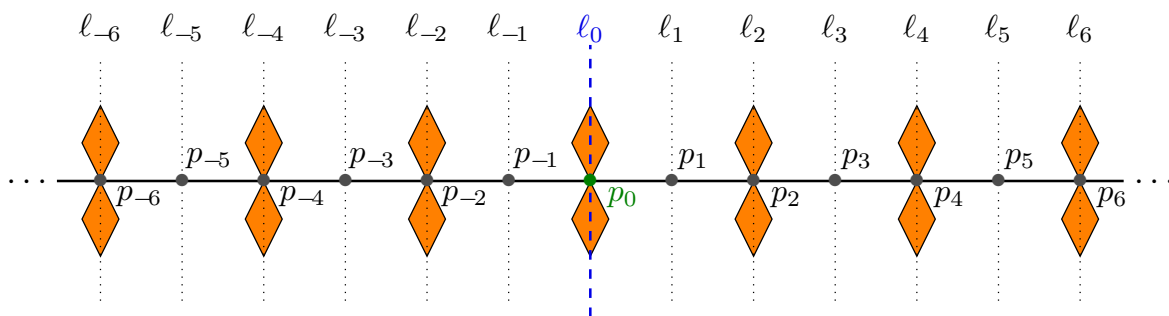
Note that the square tiles don’t actually need to be shaded. An alternate way to denote the colors of the 3×1 dominos is to underline any number with a black background. For example, using this convention, the “home state” would be written 1 2 3.

- (a) Both of these groups have 24 actions. Draw a Cayley diagram for each, with the nodes labeled by configurations. It is helpful to know that the one for **Coin**₃ can be arranged on a *truncated cube*, whose skeleton is shown below (left). A Cayley diagram for **Box**₂ can be arranged on a *truncated octahedron*, shown below (right). But the “home state” at the yellow node.



- (b) Write down a presentation for each of these groups.
- (c) Are these groups isomorphic? Justify your answer.

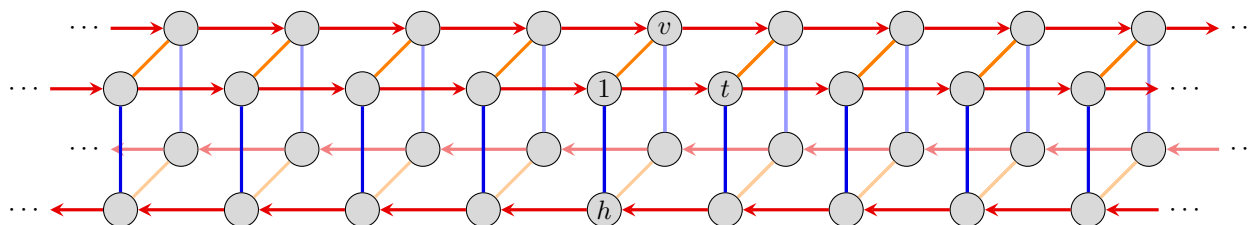
4. Consider the frieze shown below:



Let t be a minimal translation to the right, h_i a reflection across ℓ_i , and r_j a 180° rotation around p_j . Let v be the vertical reflection and $g_i = t^i v$ a glide reflection. A presentation for the frieze group is

$$\mathbf{Frz}_1 := \langle t, h, v \mid v^2 = h^2 = 1, th = ht^{-1}, tv = vt, hv = vh \rangle,$$

where $h = h_0$. A Cayley diagram is shown below.

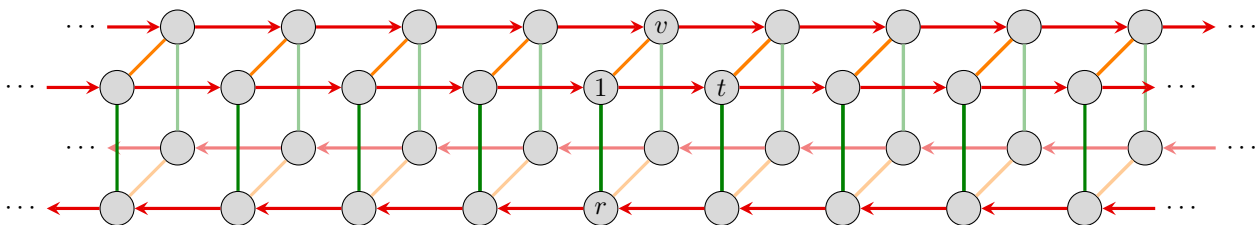


(a) Every symmetry is either a translation t^i , glide reflection g_j , rotation r_k , horizontal reflection h_ℓ , or the vertical reflection v . Label the vertices of this Cayley diagram with elements written in this form.

(b) Now, repeat the previous part, but with for the Cayley diagram for the presentation

$$\mathbf{Frz}_1 := \langle t, r, v \mid v^2 = r^2 = 1, tr = rt^{-1}, tv = vt, rv = vr \rangle,$$

where $r = r_0$. A Cayley diagram is shown below.



(c) Determine which symmetries $t^i h t^{-i}$ and $t^i r t^{-i}$ are for each $i \in \mathbb{Z}$.

