- 1. Let $A, B \leq G$ with A normalizing B (that is, $A \leq N_G(B)$, which implies that $AB \leq G$).
 - (a) Show that $B \leq AB$ and $A \cap B \leq A$.
 - (b) Show that $A/(A \cap B) \cong AB/B$. [*Hint:* Construct a homomorphism $\phi: A \to AB/B$ that has $A \cap B$ as kernel, then apply the FHT.]
- 2. Recall that the *commutator subgroup* is defined as

$$G' = \left\langle xyx^{-1}y^{-1} \mid x, y \in G \right\rangle.$$

(a) Show that G' is the intersection of all normal subgroups of G that contain the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$:

$$G' = \bigcap_{C \subseteq N \trianglelefteq G} N.$$

- (b) Show that G/G' is abelian. [*Hint*: Show that every commutator is trivial.]
- (c) For each of the groups SD_8 , $AGL_1(\mathbb{Z}_5)$, Dic_{10} , and $SL_2(\mathbb{Z}_3)$, whose subgroup lattices are shown on the supplemental material, carry out the following steps. Because all of this can be done by inspection, you need to briefly justify your answers.
 - (i) Partition the subgroups into conjugacy classes, by drawing dashed circles on the lattices.
 - (ii) The derived series of a group is defined as $G^{(0)} := G$, $G^{(1)} := G'$, $G^{(2)} := G''$, and inductively, $G^{(k)}$ is the commutator of $G^{(k-1)}$. Mark these groups on the lattice until the trivial group is reached, and determine the quotient $G^{(i)}/G^{(i+1)}$ of each successive pair.
- 3. Recall that the automorphism group of D_3 is $\operatorname{Aut}(D_3) = \langle \alpha, \beta \rangle \cong D_3$, where

$$\begin{cases} \alpha(r) = r \\ \alpha(f) = rf \end{cases} \qquad \begin{cases} \beta(r) = r^2 \\ \beta(f) = f \end{cases}$$

All of these automorphisms are *inner* (of the form $\varphi_x \colon g \mapsto x^{-1}gx$). Two Cayley diagrams for Aut (D_3) are shown below.



In this problem, we will construct analogous Cayley diagrams for $\operatorname{Aut}(D_4) \cong D_4$.

(a) For each of the Cayley diagrams of $\operatorname{Aut}(D_4)$ shown below, label the nodes with rewired copies of the Cayley diagram of $D_4 = \langle r, f \rangle$.



- (b) Repeat the previous part using the Cayley diagram of $D_4 = \langle s, t \rangle = \langle f, rf \rangle$.
- (c) Four of the eight automorphisms of D_4 are *inner*, which means they have the form $\varphi_x \colon g \mapsto x^{-1}gx$ for some $x \in D_4$. In fact, the automorphism group of D_4 is isomorphic to the semidirect product

$$\operatorname{Aut}(D_4) \cong \operatorname{Inn}(D_4) \rtimes C_2$$

of the inner automorphism group

$$\operatorname{Inn}(D_4) = \left\{ \operatorname{Id}, \varphi_r, \varphi_f, \varphi_{rf} \right\} \cong D_4 / Z(D_4) = D_4 / \langle r^2 \rangle \cong V_4$$

and the cyclic subgroup generated by an *outer automorphism* of order 2. In each of your Cayley diagrams from Parts (a) and (b), label the nodes by the corresponding automorphism written as

$$\operatorname{Aut}(D_4) = \{ \operatorname{Id}, \varphi_r, \varphi_f, \varphi_{rf}, \omega, \varphi_r \omega, \varphi_f \omega, \varphi_{rf} \omega \},\$$

where ω is the outer automorphism

$$\omega \colon D_4 \longrightarrow D_4, \qquad \alpha(r) = r, \quad \alpha(f) = rf$$

of order 4 that cyclically rotates axes of reflections of the square.

- 4. Construct each of the semidirect products below via our "inflation process".
 - (i) $D_3 \times C_2$ (ii) $D_3 \rtimes C_2$ (iii) $V_4 \rtimes C_3$.

Make sure you define the labeling maps $\theta: B \to \operatorname{Aut}(A)$. Then determine, with justification, what each group is isomorphic to. Use $D_3 = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1 \rangle$, $V_4 = \langle a, b \mid a^2 = b^2 = 1 \rangle$, and $C_n = \langle c \mid c^n = 1 \rangle$ for the individual factors, and Cayley diagrams corresponding to these generating sets.