

1. Consider the right action of $G = D_6 = \langle r, f \rangle$ on following set of 31 “binary hexagons,” where r rotates each one 60° counterclockwise, and f flips each one horizontally (i.e., across a vertical axis).

$$S = \left\{ \begin{array}{c} \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array}, \\ \\ \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline 0 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 1 & 1 & 0 \\ \hline 1 & 1 & 0 \\ \hline \end{array}, \\ \\ \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline 1 & 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 1 & 1 \\ \hline 1 & 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array}, \\ \\ \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ \hline 0 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 1 & 0 & 1 \\ \hline 1 & 0 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline 0 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & 1 & 0 \\ \hline 1 & 1 & 0 \\ \hline \end{array}, \\ \\ \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 0 & 1 & 1 \\ \hline 0 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 1 & 0 & 1 \\ \hline 1 & 0 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline 1 & 1 & 0 \\ \hline \end{array} \end{array} \right\}$$

- Draw the action diagram.
 - Construct the “fixed point table”, which has a checkmark in row g and column s if $\phi(g).s = s$.
 - Next to each $s \in S$ on your action diagram, write $\text{stab}(s)$, the stabilizer subgroup, using its generators. Which subgroups of D_6 don’t appear, and why?
 - The fixed point set $\text{fix}(g)$ of each $g \in D_6$ can be read off of the the fixed point table. What is the average size $|\text{fix}(g)|$?
 - Find $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.
 - Repeat the previous parts of this problem, but for the action of $G = D_3 = \langle r, f \rangle$ on S , where r rotates each hexagon 120° counterclockwise.
2. Suppose that G acts on S via the homomorphism $\phi: G \rightarrow \text{Perm}(S)$.
- Show that $\text{stab}(s)$ is a subgroup for all $s \in S$. Use the notational conventions that we have been using in lecture.
 - Show that the stablizers of any two elements in the same orbit are conjugate – specifically, that $\text{stab}(s.\phi(g)) = g^{-1} \text{stab}(s)g$ for all $g \in G$ and $s \in S$. This relationship is summarized by the following commutative diagram.

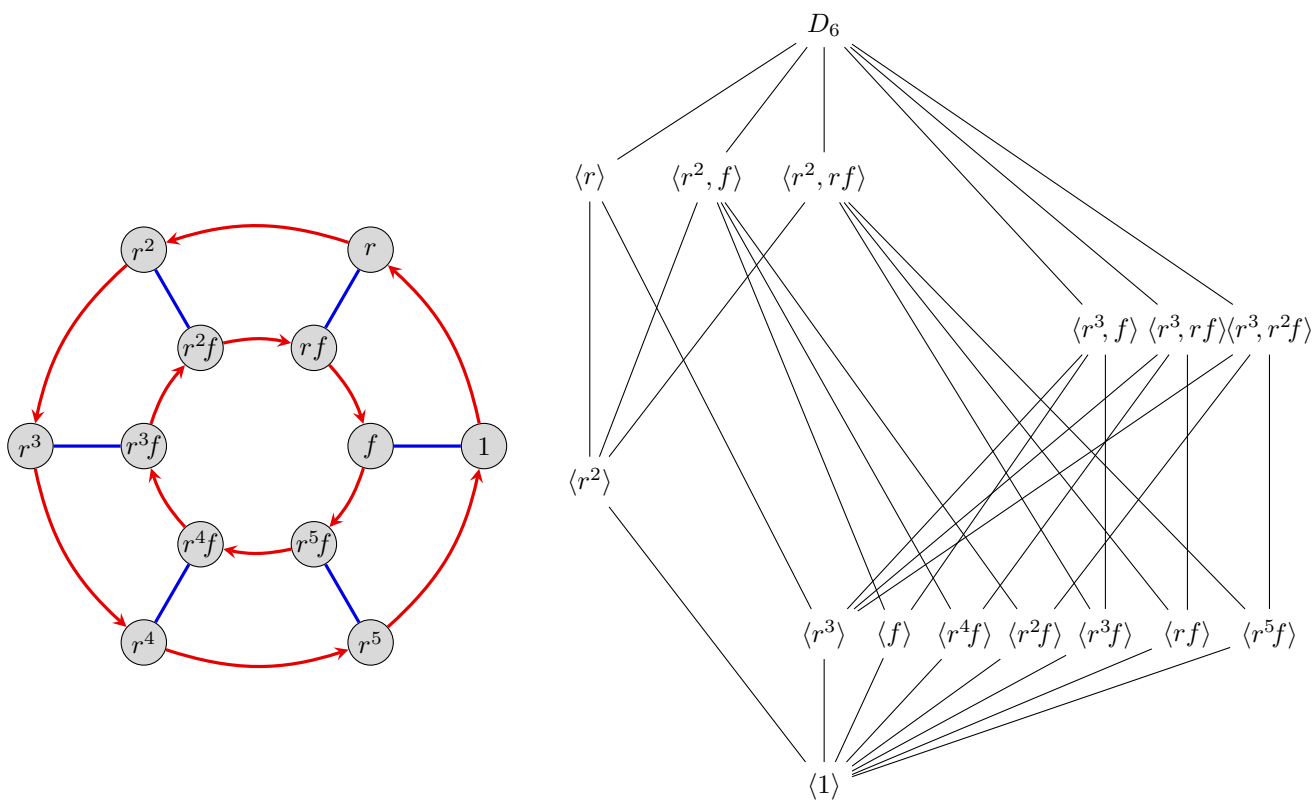
$$\begin{array}{ccc} s & \xrightarrow{\phi(x)} & s \\ \phi(g) \downarrow & & \downarrow \phi(g) \\ s' & \xrightarrow{\phi(g^{-1}xg)} & s' \end{array}$$

3. Suppose a group G of order 55 acts on a set S of size 14. Let $s \in S$ be an arbitrary element.
- What are the possible sizes of the orbit of s ?
 - What are the possible sizes of the stabilizer of s ?
 - Show that this action must have a fixed point.
 - What is the fewest number of fixed points that this action can have? Justify your answer.

4. Let $G = D_6 = \langle r, f \rangle$ act on its set $S = \{H \leq D_6\}$ of subgroups by conjugation, i.e.,

$$\phi: G \longrightarrow \text{Perm}(S), \quad \phi(g) = \text{the permutation that sends each } H \mapsto g^{-1}Hg.$$

A Cayley diagram for D_6 and its subgroup lattice are shown below.



- Construct the action diagram, and superimpose it on the subgroup lattice.
- Construct the fixed point table.
- Find $\text{stab}(H)$ for each subgroup $H \leq D_6$, and $\text{fix}(g)$ for each $g \in D_6$.
- Find $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.
- Interpret $\text{orb}(H)$, $\text{stab}(H)$, $[G : \text{stab}(H)]$, $\text{Fix}(\phi)$, $\text{Ker}(\phi)$, $\text{fix}(g)$, and the average size of $|\text{fix}(g)|$ in terms of familiar algebraic objects.