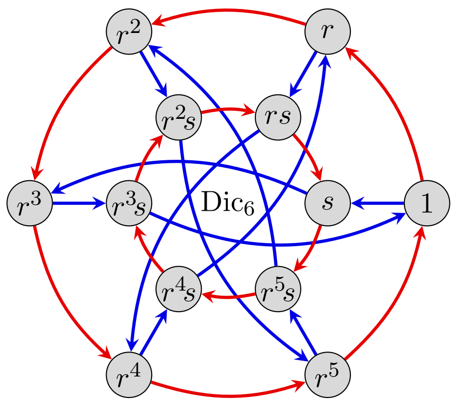
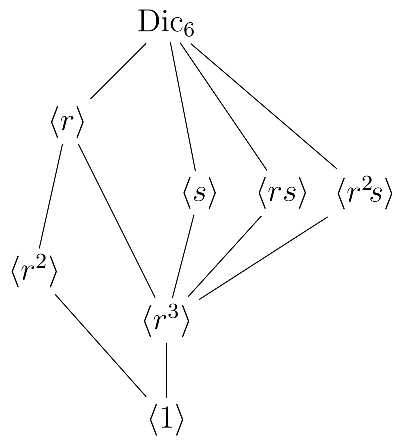
1. In this problem, we will explore the actions of the dicyclic group Dic_6 and its automorphism group on itself and its subgroups by conjugation. A Cayley diagram, subgroup lattice, and conjugacy classes are shown below.





rs	r^5s	r^3s
s	r^2s	r^4s
r^3	r^2	r^4
1	r	r^5

(a) The right action of Dic_6 on itself by conjugation is defined by the homomorphism

$$\phi\colon\ \mathrm{Dic}_6 \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

Draw the action diagram and construct the fixed point table. Find $\mathrm{stab}(s)$ for each $s \in S$, $\mathrm{fix}(g)$ for each $g \in G$, as well as $\mathrm{Ker}(\phi)$ and $\mathrm{Fix}(\phi)$.

(b) The inner automorphism group

$$\mathrm{Inn}(\mathrm{Dic}_6) \cong \mathrm{Dic}_6 / Z(\mathrm{Dic}_6) = \mathrm{Dic}_6 / \langle r^3\rangle \cong D_3$$

acts on Dic_6, and the action diagram is the same as the one in Part (a). Construct the fixed point table of this action and find $\mathrm{stab}(s)$, $\mathrm{fix}(g)$, $\mathrm{Ker}(\phi)$ and $\mathrm{Fix}(\phi)$. Then draw the subgroup lattice of $\mathrm{Inn}(\mathrm{Dic}_6) = \langle \varphi_r, \varphi_s\rangle$, where $\varphi_g\colon x \mapsto g^{-1}xg$.

(c) The automorphism group of Dic_6 is $\mathrm{Aut}(\mathrm{Dic}_6) = \langle \varphi_r, \varphi_s, \alpha\rangle$, where α is the outer automorphism defined by

$$\alpha\colon\ \mathrm{Dic}_6 \longrightarrow \mathrm{Dic}_6, \qquad \alpha(r) = r, \quad \alpha(s) = s^{-1} = r^3s,$$

that "reverses" the blue arrows. Construct the action diagram, fixed point table, and find $\mathrm{stab}(s)$, $\mathrm{fix}(g)$, $\mathrm{Ker}(\phi)$ and $\mathrm{Fix}(\phi)$.

(d) What familiar group is $\mathrm{Aut}(\mathrm{Dic}_6) = \langle \varphi_r, \varphi_s, \alpha\rangle$ isomorphic to? Construct a Cayley diagram and subgroup lattice.

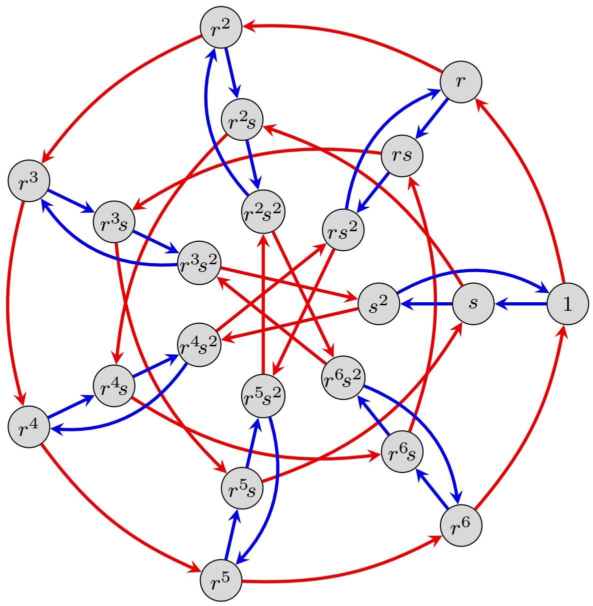
(e) The group $\mathrm{Aut}(\mathrm{Dic}_6)$ also acts on the conjugacy classes of Dic_6. Construct the action diagram, fixed point table, and find $\mathrm{stab}(s)$, $\mathrm{fix}(g)$, $\mathrm{Ker}(\phi)$ and $\mathrm{Fix}(\phi)$.

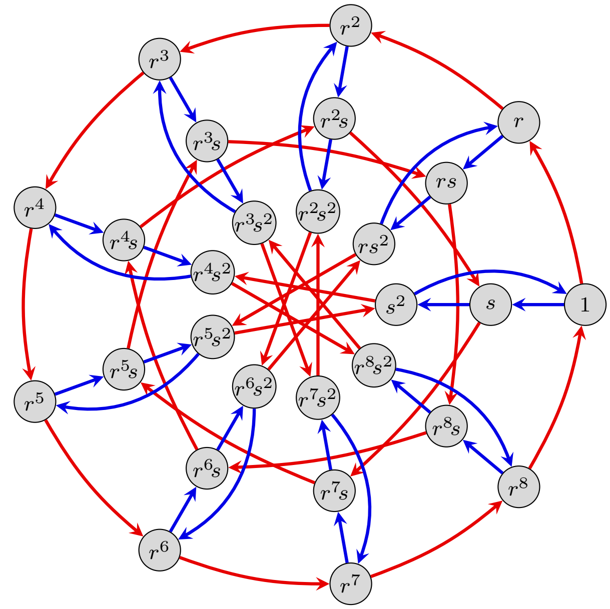
(f) The group Dic_6 acts on its subgroups by conjugation, via the homomorphism

$$\phi\colon\ \mathrm{Dic}_6 \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } H \mapsto g^{-1}Hg.$$

Construct the action diagram superimposed on the subgroup lattice. Then construct the fixed point table and find $\mathrm{stab}(s)$, $\mathrm{fix}(g)$, $\mathrm{Ker}(\phi)$ and $\mathrm{Fix}(\phi)$.

2. Carry out the following steps for the groups $C_7 \rtimes C_3$ and $C_9 \rtimes C_3$, whose Cayley diagrams are shown below.





(a) Let G act on its subgroups by conjugation. Draw the action diagram superimposed on the subgroup lattice. Construct the fixed point table, and find $\text{stab}(H)$ for each $H \leq G$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$. Which collections of subgroups arise as $\text{fix}(g)$ for some $g \in G$, and why?

(b) Let G act on the right cosets of $H = \langle s \rangle$, via the homomorphism

$$\phi \colon G \longrightarrow \text{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \mapsto Hxg.$$

Construct the action diagram, fixed point table, and find $\text{stab}(Hx)$ for each right coset, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$. Which subsets of S arise as $\text{fix}(g)$ for some $g \in G$, and why?

3. Let G be a group, not necessarily finite, and let A and B be subgroups of finite index, but not necessarily normal (in particular, we cannot assume that AB is a group).

(a) Show that even if AB is not a group, it is a disjoint union of cosets of A.

(b) Let B act on $S = AB/A = \big\{Ax \mid x \in AB\big\}$ via the homomorphism

$$\phi \colon G \longrightarrow \text{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } Ax \mapsto Axg.$$

Show that there is only one orbit.

(c) Use the orbit-stabilizer theorem to show that $[A : A \cap B] = [AB : B]$.

(d) Show that $[G : A \cap B] \leq [G : A][G : B]$. Give an explicit example of where the inequality is strict.

(e) Show that there is some $N \trianglelefteq G$ contained in both A and B with $[G : N] \leq \infty$.