1. For each of the following rings R, determine the zero divisors and the set $U(R)$ of units.

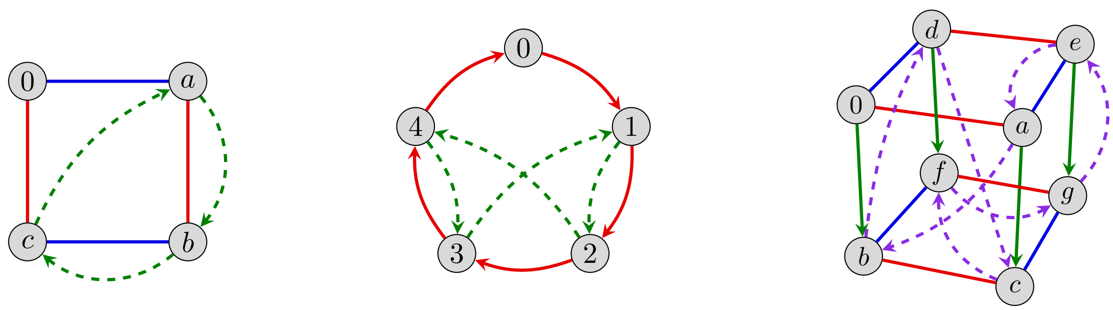
   (a) The set $\mathcal{C}^1$ of continuous real-valued functions $f\colon \mathbb{R} \to \mathbb{R}$.

   (b) The polynomial ring $\mathbb{R}[x]$.

   (c) $\mathbb{Z} \times \mathbb{Z}$, where addition and multiplication are defined componentwise.

   (d) $\mathbb{R} \times \mathbb{R}$, where addition and multiplication are defined componentwise.

2. There is a unique finite field $\mathbb{F}_q$ of order $q = p^k$ for every prime p and $k \in \mathbb{N}$. For all other $q \in \mathbb{N}$, there is no finite field of order q. For each of the fields $\mathbb{F}_4$, $\mathbb{F}_5$, and $\mathbb{F}_8$, the Cayley diagrams for addition and multiplication are shown below, overlayed on the same set of nodes. The solid arrows are the Cayley diagrams for addition and the dashed arrows are the Cayley diagrams for multiplication.



   (a) Create Cayley tables for these fields, for both addition and multiplication.

   (b) Create Cayley diagrams for the finite fields $\mathbb{F}_3$, $\mathbb{F}_7$, and $\mathbb{F}_{11}$.

3. Let I and J be ideals of a ring R.

   (a) Show that if a left ideal I of a ring R contains a unit, then $I = R$.

   (b) Show that $I + J$, $I \cap J$, and IJ are ideals of R, where

$$IJ = \{x_1y_1 + \cdots + x_ky_k \mid x_i \in I,\ y_j \in J\}.$$

   (c) If R is commutative, then the set

$$(I : J) = \{r \in R \mid rJ \subseteq I\}$$

   is called the *colon ideal* of I and J. Show that $(I : J)$ is an ideal of R.

   (d) Consider the ideals $I = 4\mathbb{Z}$ and $J = 6\mathbb{Z}$ of the ring $R = \mathbb{Z}$. Compute $I + J$, $I \cap J$, IJ, $(I : J)$, and $(J : I)$.

   (e) Repeat Part (c) for the ideals $I = m\mathbb{Z}$ and $J = n\mathbb{Z}$ of $R = \mathbb{Z}$.

4. The left ideal generated by $X \subseteq R$ is defined as

$$(X) := \bigcap \{I \mid I \text{ is a left ideal s.t. } X \subseteq I \subseteq R\}.$$

(a) Show that the left ideal generated by X is

$$(X) = \{r_1x_1 + \cdots + r_nx_n \mid n \in \mathbb{N},\ r_i \in R,\ x_i \in X\}.$$

(b) The two-sided ideal generated by $X \subseteq R$ is defined by relacing "left" with "two-sided" in the definition above. Show that this is also equal to

$$\{r_1x_1s_1 + \cdots + r_nx_ns_n \mid n \in \mathbb{N},\ r_i, s_i \in R,\ x_i \in X\}.$$

(c) Find a (non-commutative) ring R and a set X such that the left and two-sided ideals generated by X are different.

5. Prove the Fundamental homomorphism theorem (FHT) for rings: If $\phi\colon R \to S$ is a ring homomorphism, then $\mathrm{Ker}(\phi)$ is a two-sided ideal of R, and $R/\mathrm{Ker}(\phi) \cong \mathrm{Im}(\phi)$. You may assume the FHT for groups.