Chapter 7: Rings

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What is a ring?

A group is a set with a binary operation, satisfying a few basic properties.

Many algebraic structures (numbers, matrices, functions) have two binary operations.

Definition

A ring is an additive (abelian) group R with an additional binary operation (multiplication), satisfying the distributive law:

$$x(y+z) = xy + xz \qquad \text{and} \qquad (y+z)x = yx + zx \quad \forall x, y, z \in R\,.$$

Remarks

- There need not be multiplicative inverses.
- Multiplication need not be commutative (it may happen that $xy \neq yx$).

A few more definitions

If $xy = yx$ for all $x, y \in R$, then R is commutative.

If R has a multiplicative identity $1 = 1_R \neq 0$, we say that "R has identity" or "unity", or "R is a ring with 1."

A subring of R is a subset $S \subseteq R$ that is also a ring.

The two rings of order 6

The additive group $\mathbb{Z}_6$ is a ring, where multiplication is defined modulo 6.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

However, this is not the only way to add a ring structure to $(\mathbb{Z}_6, +)$.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0

All finite groups we've encountered occur naturally in some context (e.g., as matrices). Rings like the one above are somewhat "contrived".

Some rings of order 4

Consider the Klein 4-group

$$V_4 \cong \{ \underbrace{(0,0)}_{0}, \underbrace{(1,0)}_{a}, \underbrace{(0,1)}_{b}, \underbrace{(1,1)}_{c} \}.$$

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

There are 8 ways to define a multiplicative structure on this additive group. Here are 4:

×	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	c	a
c	0	c	a	b

×	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	0	0

×	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	b	0	b
c	0	c	0	c

×	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Here is another way, that can be represented with matrices:

$$\left\{ \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{0}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{a}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{b}, \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{c} \right\}.$$

×	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	0	b
c	0	c	b	a

It turns out that for any prime p, there are exactly 11 rings of order p^2.

Finite rings

In general, we'll be more interested in infinite rings.

However, let's say a few words about finite rings, mostly for fun.

n	1	2	3	4	5	6	7	8	9	10	11	12	16	32
# groups	1	1	1	2	1	2	1	5	2	2	1	5	14	51
# rings w/ 1	1	1	1	4	1	1	1	11	4	1	1	4	50	208
# rings	1	2	2	11	2	4	2	52	11	4	2	22	390	> 18590
# non-comm	0	0	0	2	0	0	0	18	2	0	0	18	228	?

Small noncommutative rings with 1 are "rare". There are

- 13 of size 16
- one each of sizes 8, 24, and 27
- and no others of order less than 32.

For distinct primes p and q, ($p \geq 3$), there are the following number of algebraic structures:

n	p	p^2	p^3	pq	p^2q
# groups	1	2	5	2	≤ 5
# rings	2	11	$3p + 50$	4	22

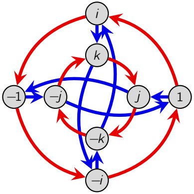
Going forward, the only fintie rings we'll typically encounter are $\mathbb{Z}_n$ and finite fields.

Some infinite rings

Examples

1. $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ are all commutative rings with 1.
2. For any ring R with 1, the set $M_n(R)$ of $n \times n$ matrices over R is a ring. It has identity $1_{M_n(R)} = I_n$ iff R has 1.
3. For any ring R, the set of functions $F = \{f \colon R \to R\}$ is a ring by defining
$$(f+g)(r) = f(r) + g(r), \qquad (fg)(r) = f(r)g(r)\,.$$
4. The set $S = 2\mathbb{Z}$ is a subring of $\mathbb{Z}$ but it does *not* have 1.
5. $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R} \right\}$ is a subring of $R = M_2(\mathbb{R})$. However, note that
$$1_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{but} \qquad 1_S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$
6. If R is a ring and x a variable, then the set
$$R[x] = \left\{ a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in R \right\}$$
is called the polynomial ring over R.

Another example: the Hamiltonians



Recall the (unit) quaternion group:

$$Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = -1,\ ij = k \rangle.$$

Allowing addition makes them into a ring $\mathbb{H}$, called the quaternions, or Hamiltonians:

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}.$$

The set $\mathbb{H}$ is isomorphic to a subring of $M_4(\mathbb{R})$, the real-valued 4×4 matrices:

$$\mathbb{H} \cong \left\{ \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} \subseteq M_4(\mathbb{R}) \,.$$

Formally, we have an embedding $\phi\colon \mathbb{H} \hookrightarrow M_4(\mathbb{R})$ where

$$\phi(i) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \phi(j) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \phi(k) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Just like with groups, we say that $\mathbb{H}$ is represented by a set of matrices.

Units and zero divisors

Informally, a ring is a set where we can add, substract, multiply, but not necessarily divide.

Definition

A unit is any $u \in R$ that has a multiplicative inverse: some $v \in R$ such that $uv = vu = 1$.

Let $U(R)$ be the set (a multiplicative group) of units of R.

An element $x \in R$ is a left zero divisor if $xy = 0$ for some $y \neq 0$. (Right zero divisors are defined analogously.)

Examples

1. Let $R = \mathbb{Z}$. The units are $U(R) = \{-1, 1\}$. There are no (nonzero) zero divisors.
2. Let $R = \mathbb{Z}_{10}$. Then 7 is a unit (and $7^{-1} = 3$) because $7 \cdot 3 = 1$. But 2 is not a unit.
3. Let $R = \mathbb{Z}_n$. A nonzero $k \in \mathbb{Z}_n$ is a unit if $\gcd(n, k) = 1$, and a zero divisor otherwise.
4. The ring $R = M_2(\mathbb{R})$ has zero divisors, such as:

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The groups of units of $M_2(\mathbb{R})$ are the invertible matrices.

Group rings

A rich family of examples of rings can be constructed from multiplicative groups.

Let G be a finite (multiplicative) group, and R a commutative ring (usually, $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$).

The group ring RG is the set of formal linear combinations of groups elements with coefficients from R. That is,

$$RG := \{a_1 g_1 + \cdots + a_n g_n \mid a_i \in R,\ g_i \in G\},$$

where multiplication is defined in the "obvious" way.

For example, let $R = \mathbb{Z}$ and $G = D_4$, and take $x = r + r^2 - 3f$ and $y = -5r^2 + rf$ in $\mathbb{Z}D_4$.

Their sum is

$$x + y = r - 4r^2 - 3f + rf,$$

and their product is

$$\begin{aligned} xy &= (r + r^2 - 3f)(-5r^2 + rf) = r(-5r^2 + rf) + r^2(-5r^2 + rf) - 3f(-5r^2 + rf) \\ &= -5r^3 + r^2 f - 5r^4 + r^3 f + 15fr^2 - 3frf = -5 - 8r^3 + 16r^2 f + r^3 f. \end{aligned}$$

Group rings

For another example, consider the group ring $\mathbb{R}Q_8$. Elements are formal sums

$$a + bi + cj + dk + e(-1) + f(-i) + g(-j) + h(-k), \qquad a, \dots, h \in \mathbb{R}.$$

Every choice of coefficients gives a different element in $\mathbb{R}Q_8$!

For example, if all coefficients are zero except $a = e = 1$, we get

$$1 + (-1) \neq 0 \in \mathbb{R}Q_8.$$

In contrast, in the Hamiltonians, $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$,

$$1 + (-1) = [1 + 0i + 0j + 0k] + [(-1) + 0i + 0j + 0k] = (1 - 1) + 0i + 0j + 0k = 0.$$

Therefore, $\mathbb{H}$ and $\mathbb{R}Q_8$ are different rings.

Remarks

- If $g \in G$ has finite order $|g| = k > 1$, then RG always has zero divisors:

$$(1 - g)(1 + g + \cdots + g^{k-1}) = 1 - g^k = 1 - 1 = 0.$$

- RG contains a subring isomorphic to R.
- the group of units $U(RG)$ contains a subgroup isomorphic to G.

Fields and division rings

Definition

A field is a commutative ring where all nonzero elements have a multiplicative inverse.

Examples of fields we've seen include $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_p$ for prime p.

Definition

A quadratic field is any field of the form

$$\mathbb{Q}(\sqrt{m}) = \left\{r + s\sqrt{m} \mid r, s \in \mathbb{Q}\right\},$$

where $m \neq 0, 1$ is a square-free integer. We say "*$\mathbb{Q}$ adjoin $\sqrt{m}$*"

Notice that this is a field because every nonzero number has a multiplicative inverse:

$$(r + s\sqrt{m})(r - s\sqrt{m}) = r^2 - s^2 m, \qquad (r + s\sqrt{m})^{-1} = \frac{r - s\sqrt{m}}{r^2 - s^2 m}.$$

If we drop the commutative requirement, the result is called a skew field, or division ring.

The Hamiltonians $\mathbb{H}$ are a division ring that is not a field.

Integral domains

Definition

An integral domain is a commutative ring with 1 and with no (nonzero) zero divisors.

An integral domain is a "field without inverses".

A field is just a commutative division ring. Moreover:

$$\text{fields} \subsetneq \text{division rings}, \qquad \text{fields} \subsetneq \text{integral domains}.$$

Examples

- Rings that are not integral domains: $\mathbb{Z}_n$ (composite n), $2\mathbb{Z}$, $M_n(\mathbb{R})$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{H}$.
- Integral domains that are not fields (or even division rings): $\mathbb{Z}$, $\mathbb{Z}[x]$, $\mathbb{R}[x]$, $\mathbb{R}[[x]]$ (formal power series).

The ring "$\mathbb{Z}$ *adjoin* $\sqrt{m}$," defined as

$$\mathbb{Z}[\sqrt{m}] = \{a + b\sqrt{m} \mid a, b \in \mathbb{Z}\},$$

is an integral domain, but not a field.

Cancellation

When doing basic algebra, we often take for granted basic properties such as cancellation:

$$ax = ay \implies x = y.$$

However, *this need not hold in all rings!*

Examples where cancellation fails

- In $\mathbb{Z}_6$, note that $2 = 2 \cdot 1 = 2 \cdot 4$, but $1 \neq 4$.
- In $M_2(\mathbb{R})$, note that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.

However, everything works fine as long as there aren't any (nonzero) zero divisors.

Proposition

Let R be an integral domain and $a \neq 0$. If $ax = ay$ for some $x, y \in R$, then $x = y$.

Proof

If $ax = ay$, then $ax - ay = a(x - y) = 0$.

Since $a \neq 0$ and R has no (nonzero) zero divisors, then $x - y = 0$. □

Finite integral domains

Remark

If R is an integral domain and $0 \neq a \in R$ and $k \in \mathbb{N}$, then $a^k \neq 0$. □

Theorem

Every finite integral domain is a field.

Proof

Suppose R is a finite integral domain and $0 \neq a \in R$. It suffices to show that a has a multiplicative inverse.

Consider the infinite sequence $a, a^2, a^3, a^4, \dots,$ which must repeat.

Find $i > j$ with $a^i = a^j$, which means that

$$0 = a^i - a^j = a^j(a^{i-j} - 1).$$

Since R is an integral domain and $a^j \neq 0$, then $a^{i-j} = 1$.

Thus, $a \cdot a^{i-j-1} = 1$. □

Ideals

In group theory, we can quotient out by a subgroup if and only if it is normal.

The analogue of this for rings are (two-sided) ideals.

Definition

A subring $I \subseteq R$ is a left ideal if

$$rx \in I \qquad \text{for all } r \in R \text{ and } x \in I.$$

Right ideals, and two-sided ideals are defined similarly.

If R is commutative, then all left (or right) ideals are two-sided.

We use the term ideal and two-sided ideal synonymously, and write $I \trianglelefteq R$.

Examples

- $n\mathbb{Z} \trianglelefteq \mathbb{Z}$.
- If $R = M_2(\mathbb{R})$, then $I = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} : a, c \in \mathbb{R} \right\}$ is a left, but *not* a right ideal of R.
- The set $\mathsf{Sym}_n(\mathbb{R})$ of symmetric $n \times n$ matrices is a subring of $M_n(\mathbb{R})$, but *not* an ideal.
- The set $\mathbb{Z}$ is a subring of $\mathbb{Z}[x]$ but not an ideal.

Ideals

Remark

If an ideal I of R contains 1, then $I = R$.

Proof

Suppose $1 \in I$, and take an arbitrary $r \in R$.

Then $r1 \in I$, and so $r1 = r \in I$. Therefore, $I = R$. □

We can modify the above result to show that if I contains *any* unit, then $I = R$. (HW)

Let's compare the concept of a normal subgroup to that of an ideal:

- normal subgroups are characterized by being invariant under conjugation:

$$H \leq G \text{ is normal} \quad \text{iff} \quad ghg^{-1} \in H \text{ for all } g \in G,\ h \in H.$$

- (left) ideals of rings are characterized by being invariant under (left) multiplication:

$$I \subseteq R \text{ is a (left) ideal} \quad \text{iff} \quad rx \in I \text{ for all } r \in R,\ x \in I.$$

Ideals generated by sets

Definition

The left ideal generated by a set $X \subset R$ is defined as:

$$(X) := \bigcap \{I : \ I \text{ is a left ideal s.t. } X \subseteq I \subseteq R\}.$$

This is the smallest left ideal containing X.

There are analogous definitions by replacing "left" with "right" or "two-sided".

Recall the two ways to define the subgroup $\langle X \rangle$ generated by a subset $X \subseteq G$:

- "*Bottom up*": As the set of all finite products of elements in X;
- "*Top down*": As the intersection of all subgroups containing X.

Proposition (HW)

Let R be a ring with 1. The (left, right, two-sided) ideal generated by $X \subseteq R$ is:

- Left: $\{r_1x_1 + \cdots + r_nx_n : n \in \mathbb{N},\ r_i \in R,\ x_i \in X\}$,
- Right: $\{x_1r_1 + \cdots + x_nr_n : n \in \mathbb{N},\ r_i \in R,\ x_i \in X\}$,
- Two-sided: $\{r_1x_1s_1 + \cdots + r_nx_ns_n : n \in \mathbb{N},\ r_i, s_i \in R,\ x_i \in X\}$.

Ideals generated by sets

As we did with groups, if $S = \{x\}$, we can write (x) rather than $(\{x\})$, etc.

Let's see some examples of ideals in $R = \mathbb{Z}[x]$.

$$(x) = \big\{xf(x) \mid f \in \mathbb{Z}[x]\big\} = \big\{a_n x^n + \cdots + a_1 x \mid a_i \in \mathbb{Z}\big\}.$$

$$(2) = \big\{2f(x) \mid f \in \mathbb{Z}[x]\big\} = \big\{2a_n x^n + \cdots + 2a_1 x + 2a_0 \mid a_i \in \mathbb{Z}\big\}.$$

$$(x, 2) = \big\{xf(x) + 2g(x) \mid f, g \in \mathbb{Z}[x]\big\} = \big\{a_n x^n + \cdots + a_1 x + 2a_0 \mid a_i \in \mathbb{Z}\big\}.$$

Notice that we have

$$(x) \subsetneq (x, 2) \subsetneq R, \qquad \text{and} \qquad (2) \subsetneq (x, 2) \subsetneq R.$$

The ideal $(x, 2)$ is said to be maximal, because there is nothing "between" it and R.

Question

How different would these ideals be in the ring $R = \mathbb{Q}[x]$?

Ideals and quotients

Since an ideal I of R is an additive subgroup (and hence normal), then:

- $R/I = \{x + I \mid x \in R\}$ is the set of cosets of I in R;
- R/I is a quotient group; with the binary operation (addition) defined as

$$(x + I) + (y + I) := x + y + I.$$

It turns out that if I is also a two-sided ideal, then we can make R/I into a ring.

Proposition

If $I \subseteq R$ is a (two-sided) ideal, then R/I is a ring (called a quotient ring), where multiplication is defined by

$$(x + I)(y + I) := xy + I.$$

Proof

We need to show this is well-defined. Suppose $x + I = r + I$ and $y + I = s + I$. This means that $x - r \in I$ and $y - s \in I$.

It suffices to show that $xy + I = rs + I$, or equivalently, $xy - rs \in I$:

$$xy - rs = xy - ry + ry - rs = (x - r)y + r(y - s) \in I.$$

Motivation (spoilers!)

Many of the big ideas from group homomorphisms carry over to ring homomorphisms.

Group theory

- The quotient group G/N exists iff N is a normal subgroup.
- A homomorphism is a structure-preserving map: $f(x * y) = f(x) * f(y)$.
- The kernel of a homomorphism is a normal subgroup: $\text{Ker}(\phi) \trianglelefteq G$.
- For every normal subgroup $N \trianglelefteq G$, there is a natural quotient homomorphism $\phi \colon G \to G/N$, $\phi(g) = gN$.
- There are four standard isomorphism theorems for groups.

Ring theory

- The quotient ring R/I exists iff I is a two-sided ideal.
- A homomorphism is a structure-preserving map: $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$.
- The kernel of a homomorphism is a two-sided ideal: $\text{Ker}(\phi) \trianglelefteq R$.
- For every two-sided ideal $I \trianglelefteq R$, there is a natural quotient homomorphism $\phi \colon R \to R/I$, $\phi(r) = r + I$.
- There are four standard isomorphism theorems for rings.

Ring homomorphisms

Definition

A ring homomorphism is a function $f\colon R \to S$ satisfying

$$f(x+y) = f(x) + f(y) \qquad \text{and} \qquad f(xy) = f(x)f(y) \quad \text{for all } x, y \in R.$$

A ring isomorphism is a homomorphism that is bijective.

The kernel $f\colon R \to S$ is the set $\mathrm{Ker}(f) := \{x \in R \mid f(x) = 0\}$.

Examples

1. The ring homomorphism $\phi\colon \mathbb{Z} \to \mathbb{Z}_n$ sending $k \mapsto k \pmod n$ has $\mathrm{Ker}(\phi) = n\mathbb{Z}$.
2. For a fixed real number $\alpha \in \mathbb{R}$, the "evaluation function"

   $$\phi\colon \mathbb{R}[x] \longrightarrow \mathbb{R}\,, \qquad \phi\colon p(x) \longmapsto p(\alpha)$$

   is a homomorphism. The kernel consists of all polynomials that have α as a root.
3. The following is a homomorphism, for the ideal $I = (x^2 + x + 1)$ in $\mathbb{Z}_2[x]$:

   $$\phi\colon \mathbb{Z}_2[x] \longrightarrow \mathbb{Z}_2[x]/I, \qquad f(x) \longmapsto f(x) + I\,.$$

Ring homomorphisms

Proposition

The kernel of a ring homomorphism $\phi \colon R \to S$ is a two-sided ideal.

Proof

We know that $\mathrm{Ker}(\phi)$ is an additive subgroup of R.

We must show that it's a subring, and an ideal.

Subring: Let $k_1, k_2 \in \mathrm{Ker}(\phi)$. Then

$$\phi(k_1 k_2) = \phi(k_1)\phi(k_2) = 0 \cdot 0 = 0,$$

and so $k_1 k_2 \in \mathrm{Ker}(\phi)$. ✓

Left ideal: Let $k \in \mathrm{Ker}(\phi)$ and $r \in R$. Then

$$\phi(rk) = \phi(r)\phi(k) = r \cdot 0 = 0,$$

and so $rk \in \mathrm{Ker}(\phi)$. ✓

Showing that $\mathrm{Ker}(\phi)$ is a right ideal is analogous. □

The isomorphism theorems for rings

All of the isomorphism theorems for groups have analogues for rings.

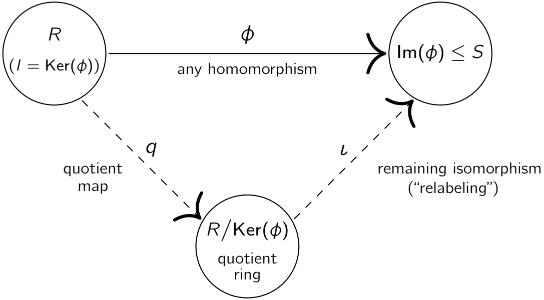
- Fundamental homomorphism theorem: "*All homomorphic images are quotients*"
- Correspondence theorem: Characterizes "*subrings and ideals of quotients*"
- Freshman theorem: Characterizes "*quotients of quotients*"
- Diamond isomorphism theorem: characterizes "*quotients of a sum*"

Since a ring is an abelian group with extra structure, we often don't have to prove these from scratch.

The FHT for rings: all homomorphic images are quotients

Fundamental homomorphism theorem for rings

If $\phi\colon R \to S$ is a ring homomorphism, then $\text{Ker}(\phi)$ is an ideal and $\text{Im}(\phi) \cong R/\text{Ker}(\phi)$.



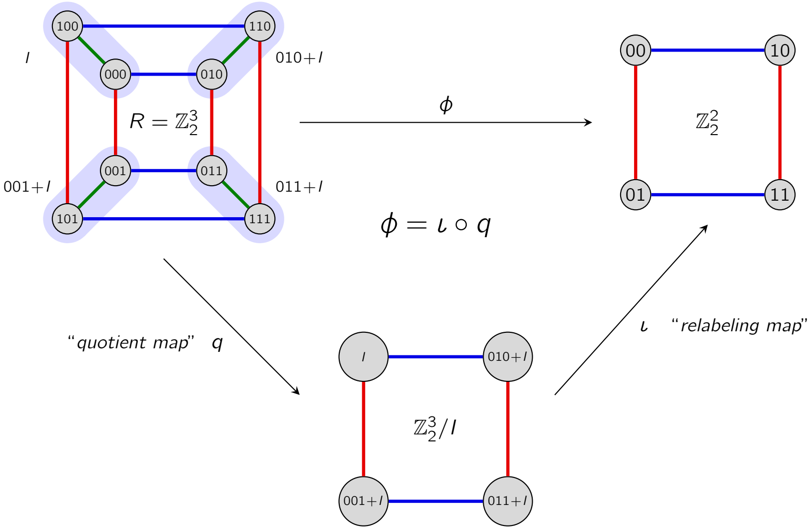
Proof (HW)

The statement holds for the underlying additive group R. Thus, it remains to show that $\text{Ker}(\phi)$ is a (two-sided) ideal, and the following relabeling map is a ring homomorphism:

$$\iota\colon R/I \longrightarrow \text{Im}(\phi)\,, \qquad \iota(r+I) = \phi(r)\,.$$

The FHT for rings

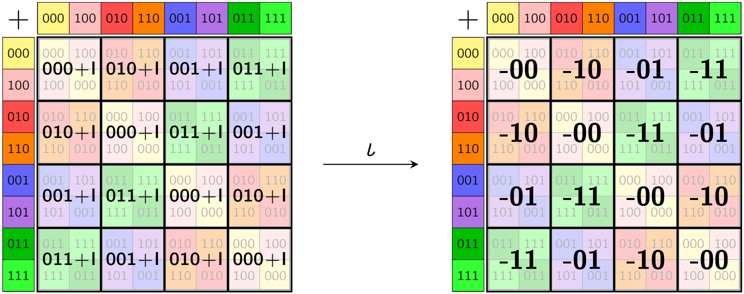
Consider the ring homomorphism $\phi\colon \mathbb{Z}_2^3 \longrightarrow \mathbb{Z}_2^2, \qquad \phi\colon abc \longmapsto bc.$



The FHT for rings

Consider the ring homomorphism $\phi\colon \mathbb{Z}_2^3 \longrightarrow \mathbb{Z}_2^2, \qquad \phi\colon abc \longmapsto bc.$

By the FHT for groups, we know that $\mathbb{Z}_2^3/\operatorname{Ker}(\phi) \cong \operatorname{Im}(\phi) = \mathbb{Z}_2^2$, as (additive) groups.



The image is isomorphic to the Klein 4-group

$$V_4 \cong \{\underbrace{(0,0)}_{0}, \underbrace{(1,0)}_{a}, \underbrace{(0,1)}_{b}, \underbrace{(1,1)}_{c}\}.$$

	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

+	00	10	01	11
00	00	10	01	11
10	10	00	11	01
01	01	11	00	10
11	11	01	10	00

The FHT theorem for rings says that ι also preserves the multiplicative structure of R/I.

The FHT for rings

Consider the ring homomorphism $\phi\colon \mathbb{Z}_2^3 \longrightarrow \mathbb{Z}_2^2$, $\qquad \phi\colon abc \longmapsto bc$.

The following Cayley tables show how ι preserves the multiplicative structure:

$$\iota\big((r+I)(s+I)\big) = \iota(rs+I).$$

×	000	100	010	110	001	101	011	111
000	000	000	000	000	000	000	000	000
100	000	100	000	100	000	100	000	100
010	000	000	010	010	000	000	010	010
110	000	100	010	110	000	100	010	110
001	000	000	000	000	001	001	001	001
101	000	100	000	100	001	101	001	101
011	000	000	010	010	001	001	011	011
111	000	100	010	110	001	101	011	111

×	000, 100	010, 110	001, 101	011, 111
000, 100	000+I	000+I	000+I	000+I
010, 110	000+I	010+I	000+I	010+I
001, 101	000+I	000+I	001+I	001+I
011, 111	000+I	010+I	001+I	011+I

$\xrightarrow{\ \iota\ }$

×	000, 100	010, 110	001, 101	011, 111
000, 100	-00	-00	-00	-00
010, 110	-00	-10	-00	-10
001, 101	-00	-00	-01	-01
011, 111	-00	-10	-01	-11

This quotient ring is isomorphic to

$$\Bigg\{ \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{0}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{a}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{b}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{c} \Bigg\}.$$

×	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

×	00	10	01	11
00	00	00	00	00
10	00	10	00	10
01	00	00	01	01
11	00	10	01	11

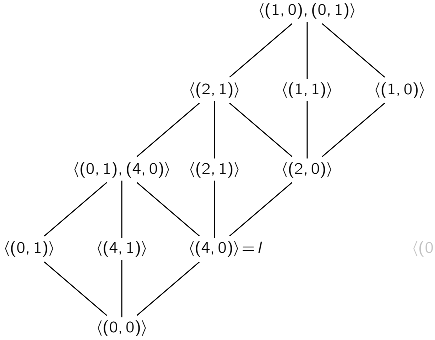
The correspondence theorem: subrings of quotients

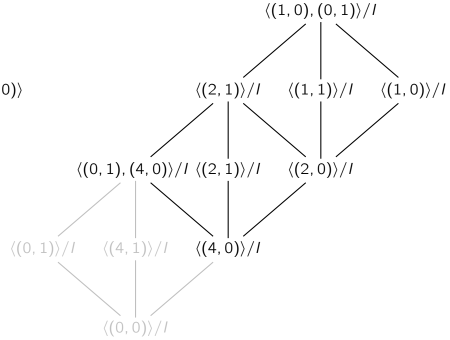
Correspondence theorem

Let I be an ideal of R. There is a bijective correspondence between subrings of R/I and subrings of R that contain I.

Moreover every ideal of R/I has the form J/I, for some ideal J satisfying $I \subseteq J \subseteq R$.

Here is an example for the ring $R = \mathbb{Z}_8 \times \mathbb{Z}_2$:





Maximal ideals and simple rings

Define a maximal normal subgroup M of G is one for which there are no normal subgroups properly between them.

Formally, we can write this as

$$M \leq N \leq G, \quad \text{and} \quad M, N \trianglelefteq G \qquad \Longrightarrow \qquad N = M, \text{ or } N = G.$$

By the correspondence theorem, M is a maximal normal subgroup iff G/M is simple.

We can define analogous terms for rings.

Definition

A (proper) ideal I of R is maximal if $I \subseteq J \subseteq R$ holds implies $J = I$ or $J = R$.

A ring R is simple if its only (two-sided) ideals are 0 and R.

The following is immediate by the correspondence theorem.

Remark

An ideal M of R is maximal iff R/M is simple.

Maximal ideals and simple rings

Simple rings have no nontrivial proper ideals. Proper ideals cannot contain units.

In a field, *every* nonzero element is a unit. Therefore, fields have no nontrivial proper ideals.

Proposition

A commutative ring R is simple iff it is a field.

Proof

"$\Rightarrow$": Assume R is simple. Then $(a) = R$ for any nonzero $a \in R$.

Thus, $1 \in (a)$, so $1 = ba$ for some $b \in R$, so $a \in U(R)$ and R is a field. ✓

"$\Leftarrow$": Let $I \subseteq R$ be a nonzero ideal of a field R. Take any nonzero $a \in I$.

Then $a^{-1}a \in I$, and so $1 \in I$, which means $I = R$. ✓ □

Theorem

Let R be a commutative ring with 1. The following are equivalent for an ideal $I \subseteq R$.

(i) I is a maximal ideal;

(ii) R/I is simple;

(iii) R/I is a field.

Examples of maximal ideals

In a commutative ring, an ideal $M \neq 0$ is a maximal iff R/M is a field.

1. The maximal ideals of $R = \mathbb{Z}$ are of the form $M = (p)$, where p is prime. The quotient field is $\mathbb{Z}/(p) \cong \mathbb{Z}_p$.

2. The maximal ideals of $R = \mathbb{Z}[x]$ are of the form

$$(x, p) = \{xf(x) + p \cdot g(x) \mid f, g \in \mathbb{Z}[x]\} = \{a^n x^n + \cdots + a_1 x + pa_0 \mid a_i \in \mathbb{Z}\}.$$

   In the quotient field, "$x := 0$" and "$p := 0$", and so

$$\mathbb{Z}[x]/(x, p) = \{a_0 + M \mid a_0 = 0, \ldots, p-1\} \cong \mathbb{Z}_p.$$

3. Let $R = \mathbb{Q}[x]$. The ideal

$$(x) = \{xf(x) \mid f \in \mathbb{Q}[x]\} = \{a^n x^n + \cdots + a_1 x \mid a_i \in \mathbb{Z}\}$$

   is maximal. In the quotient field, "$x := 0$", and so

$$\mathbb{Q}[x]/(x) = \{a_0 + M \mid a_0 \in \mathbb{Q}\} \cong \mathbb{Q}.$$

4. In the multivariant ring $R = F[x, y]$ over a field, the ideal

$$I = (x, y) = \{x \cdot f(x, y) + y \cdot g(x, y) \mid f, g \in R\}$$

   of all polynomials with no constant term is maximal. The quotient field is $R/I \cong F$.

Finite fields

We've already seen that:

- $\mathbb{Z}_p$ is a field if p is prime
- every finite integral domain is a field.

But *what do these "other" finite fields look like?*

Let $R = \mathbb{Z}_2[x]$. (Note: we can ignore all negative signs.)

The polynomial $f(x) = x^2 + x + 1$ is irreducible over $\mathbb{Z}_2$ because it does not factor as a product $f(x) = g(x)h(x)$ of lower-degree terms. (Note that $f(0) = f(1) = 1 \neq 0$.)

Consider the ideal $I = (x^2 + x + 1)$, the set of multiples of $x^2 + x + 1$.

In the quotient ring R/I, we have the relation $x^2 + x + 1 = 0$, or equivalently,

$$x^2 = -x - 1 = x + 1.$$

The quotient has only 4 elements:

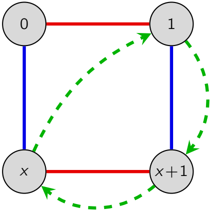
$$0 + I, \qquad 1 + I, \qquad x + I, \qquad (x+1) + I.$$

As with the quotient group (or ring) $\mathbb{Z}/n\mathbb{Z}$, we usually drop the "I", and just write

$$R/I = \mathbb{Z}_2[x]/(x^2 + x + 1) \cong \{0,\ 1,\ x,\ x+1\}.$$

Finite fields

Here are the Cayley graph and Cayley tables for $R/I = \mathbb{Z}_2[x]/(x^2+x+1)$:



$+$	0	1	x	$x+1$
0	0	1	x	$x+1$
1	1	0	$x+1$	x
x	x	$x+1$	0	1
$x+1$	$x+1$	x	1	0

$\times$	1	x	$x+1$
1	1	x	$x+1$
x	x	$x+1$	1
$x+1$	$x+1$	1	x

Theorem

There exists a finite field $\mathbb{F}_q$ of order q, which is unique up to isomorphism, iff $q = p^n$ for some prime p. If $n > 1$, then this field is isomorphic to the quotient ring

$$\mathbb{Z}_p[x]/(f),$$

where f is *any* irreducible polynomial of degree n.

Much of the error correcting techniques in coding theory are built using mathematics over $\mathbb{F}_{2^8} = \mathbb{F}_{256}$. This is what allows DVDs to play despite scratches.

Existence of maximal ideals

In a finite ring, it is clear that every ideal is contained in a maximal ideal.

To show this for infinite rings, we need the following, which is equivalent to the axiom of choice from set theory.

Zorn's lemma

If $\mathcal{P} \neq \emptyset$ is a poset in which every chain has an upper bound, then $\mathcal{P}$ has a maximal element.

Proposition

If R is a ring with 1, then every ideal $I \neq R$ is contained in a maximal ideal M.

Proof

Let $\mathcal{P} = \{J \leq R \mid I \subseteq J \subsetneq R\}$, ordered by inclusion.

Every chain $\mathcal{C}$ has a maximal element, $L_{\mathcal{C}} = \bigcup_{J \in \mathcal{C}} J$, and hence an upper bound.

By Zorn's lemma, there is some maximal element M in $\mathcal{P}$, which is a maximal ideal.

The freshman theorem: quotients of quotients

The correspondence theorem characterizes the subring structure of the quotient R/J.

Every subring of R/I is of the form J/I, where $I \leq J \leq R$.

Moreover, if $J \trianglelefteq R$ is an ideal, then $J/I \trianglelefteq R/I$. In this case, we can ask:

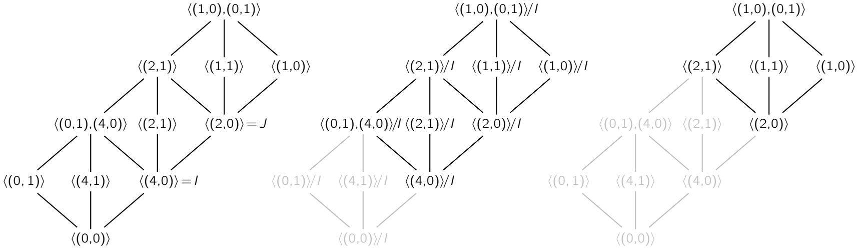
What is the quotient ring $(R/I)/(J/I)$ isomorphic to?

Freshman theorem

Suppose R is a ring with ideals $I \subseteq J$. Then J/I is an ideal of R/I and

$$(R/I)/(J/I) \cong R/J.$$

Here is an example for the ring $R = \mathbb{Z}_8 \times \mathbb{Z}_2$:



The freshman theorem: quotients of quotients

For another visualization, consider $R = \mathbb{Z}_6 \times \mathbb{Z}_4$ and write elements as strings.

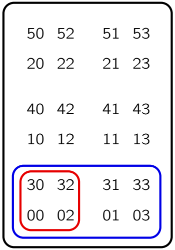
Consider the ideals $J = \langle 30, 02 \rangle \cong V_4$ and $I = \langle 30, 01 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

Notice that $I \leq J \leq R$, and $I = J \cup (01 + J)$, and

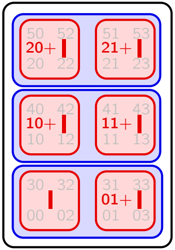
$$R/I = \big\{ I,\ 01{+}I,\ 10{+}I,\ 11{+}I,\ 20{+}I,\ 21{+}I \big\}, \qquad J/I = \{ I,\ 01{+}I \}$$

$$R/J = \Big\{ I \cup (01{+}I),\ (10{+}I) \cup (11{+}I),\ (20{+}I) \cup (21{+}I) \Big\}$$

$$(R/I)/(J/I) = \Big\{ \{I,\ 01{+}I\},\ \{10{+}I,\ 11{+}I\},\ \{20{+}I,\ 21{+}I\} \Big\}.$$

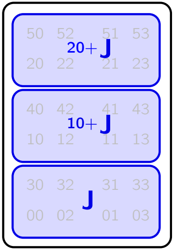


$I \leq J \leq R$



R/I consists of 6 cosets

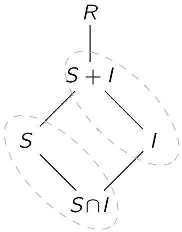
$J/I = \{I,\ 01{+}I\}$



R/J consists of 3 cosets

$(R/I)/(J/I) \cong R/J$

The diamond isomorphism theorem: quotients of sums



Diamond isomorphism theorem

Suppose S is a subring and I an ideal of R. Then

(i) The sum $S + I = \{s + i \mid s \in S,\ i \in I\}$ is a subring of R and the intersection $S \cap I$ is an ideal of S.

(ii) The following quotient rings are isomorphic:

$$(S + I)/I \cong S/(S \cap I)\,.$$

Proof (sketch)

$S + I$ is an additive subgroup, and it's closed under multiplication because

$$s_1, s_2 \in S,\ i_1, i_2 \in I \quad \Longrightarrow \quad (s_1 + i_1)(s_2 + i_2) = \underbrace{s_1 s_2}_{\in S} + \underbrace{s_1 i_2 + i_1 s_2 + i_1 i_2}_{\in I} \in S + I.$$

Showing $S \cap I$ is an ideal of S is straightforward (homework exercise).

We already know that $(S + I)/I \cong S/(S \cap I)$ as additive groups.

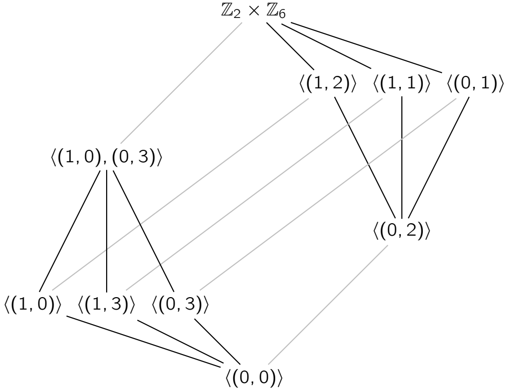
One explicit isomorphism is $\phi\colon s + (S \cap I) \mapsto s + I$. It is easy to check that $\phi\colon 1 \mapsto 1$ and ϕ preserves products. □

The diamond isomorphism theorem: quotients of products by factors

Let $R = \mathbb{Z}_2 \times \mathbb{Z}_6$, and consider the subring $S = \langle (1,0), (0,3) \rangle$ and ideal $I = \langle (0,2) \rangle$.

Then $R = I + J$, and $I \cap J = \langle (0,0) \rangle$.

Let's interpret the diamond theorem $(S+I)/I \cong S/S \cap I$ in terms of the subgroup lattice.



Prime ideals

Definition

Let R be a commutative ring. An ideal $P \subset R$ is prime if $ab \in P$ implies either $a \in P$ or $b \in P$.

Note that $p \in \mathbb{N}$ is a prime number iff $p = ab$ implies either $a = p$ or $b = p$.

Examples

1. The ideal (n) of $\mathbb{Z}$ is a prime ideal iff n is a prime number (possibly $n = 0$).
2. In the polynomial ring $\mathbb{Z}[x]$, the ideal $I = (2, x)$ is a prime ideal. It consists of all polynomials whose constant coefficient is even.

Theorem

An ideal $P \subseteq R$ is prime iff R/P is an integral domain.

The proof is straightforward (HW). Since fields are integral domains, the following is immediate:

Corollary

In a commutative ring, every maximal ideal is prime.

Divisibility and factorization

A ring is in some sense, a generalization of the familiar number systems like $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, where we are allowed to add, subtract, and multiply.

Two key properties about these structures are:

- multiplication is commutative,
- there are no (nonzero) zero divisors.

Blanket assumption

Henceforth, unless explicitly mentioned otherwise, R is assumed to be an integral domain, and we will define $R^* := R \setminus \{0\}$.

The integers have several basic properties that we usually take for granted:

- every nonzero number can be factored uniquely into primes;
- any two numbers have a unique greatest common divisor and least common multiple;
- there is a Euclidean algorithm, which can find the gcd of two numbers.

Surprisingly, these need not always hold in integrals domains! We would like to understand this better.

Divisibility

Definition

If $a, b \in R$, say that a divides b, or b is a multiple of a if $b = ac$ for some $c \in R$. We write $a \mid b$.

If $a \mid b$ and $b \mid a$, then a and b are associates, written $a \sim b$.

Examples

- In $\mathbb{Z}$: n and $-n$ are associates.
- In $\mathbb{R}[x]$: $f(x)$ and $c \cdot f(x)$ are associates for any $c \neq 0$.
- The only associate of 0 is itself.
- The associates of 1 are the units of R.

Proposition (HW)

Two elements $a, b \in R$ are associates if and only if $a = bu$ for some unit $u \in U(R)$.

This defines an equivalence relation on R, and partitions R into equivalence classes.

Irreducibles and primes

Note that units divide everything: if $b \in R$ and $u \in U(R)$, then $u \mid b$.

Definition

If $b \notin U(R)$ and its only divisors are units and associates of b, then b is irreducible.

An element $p \in R$ is prime if p is not a unit, and $p \mid ab$ implies $p \mid a$ or $p \mid b$.

Proposition

If $0 \neq p \in R$ is prime, then p is irreducible.

Proof

Suppose p is not irreducible. Then $p = ab$ with $a, b \notin U(R)$.

Then (wlog) $p \mid a$, so $a = pc$ for some $c \in R$. Now,

$$p = ab = (pc)b = p(cb)\,.$$

This means that $cb = 1$, and thus $b \in U(R)$. Therefore, p is prime. □

Irreducibles and primes

Caveat: Irreducible $\not\Rightarrow$ prime

Consider the ring $R_{-5} := \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$.

$$3 \mid (2+\sqrt{-5})(2-\sqrt{-5}) = 9 = 3 \cdot 3\,,$$

but $3 \nmid 2+\sqrt{-5}$ and $3 \nmid 2-\sqrt{-5}$.

Thus, 3 is irreducible in R_{-5} but *not* prime.

When irreducibles fail to be prime, we can lose nice properties like unique factorization.

Things can get really bad: not even the *lengths* of factorizations into irreducibles need be the same!

For example, consider the ring $R = \mathbb{Z}[x^2, x^3]$. Then

$$x^6 = x^2 \cdot x^2 \cdot x^2 = x^3 \cdot x^3.$$

The element $x^2 \in R$ is not prime because $x^2 \mid x^3 \cdot x^3$ yet $x^2 \nmid x^3$ in R (note: $x \notin R$).

Principal ideal domains

Fortunately, there is a type of ring where such "bad things" don't happen.

Definition

An ideal generated by a single element $a \in R$, denoted $I = (a)$, is called a principal ideal.

If every ideal of R is principal, then R is a principal ideal domain (PID).

Examples

The following are all PIDs (stated without proof):

- The ring of integers, $\mathbb{Z}$.
- Any field F.
- The polynomial ring $F[x]$ over a field.

As we will see shortly, PIDs are "nice" rings. Here are some properties they enjoy:

- pairs of elements have a "greatest common divisor" & "least common multiple"
- irreducible $\Rightarrow$ prime
- Every element factors uniquely into primes.

Greatest common divisors & least common multiples

Proposition

If $I \subseteq \mathbb{Z}$ is an ideal, and $a \in I$ is its smallest positive element, then $I = (a)$.

Proof

Pick any positive $b \in I$. Write $b = aq + r$, for $q, r \in \mathbb{Z}$ and $0 \leq r < a$.

Then $r = b - aq \in I$, so $r = 0$. Therefore, $b = qa \in (a)$. $\square$

Definition

A common divisor of $a, b \in R$ is an element $d \in R$ such that $d \mid a$ and $d \mid b$.

Moreover, d is a greatest common divisor (GCD) if $c \mid d$ for all other common divisors c of a and b.

A common multiple of $a, b \in R$ is an element $m \in R$ such that $a \mid m$ and $b \mid m$.

It's a least common multiple (LCM) if $m \mid n$ for all other common multiples n of a and b.

Nice properties of PIDs

Proposition

If R is a PID, then any $a, b \in R^*$ have a GCD, $d = \mathbf{gcd}(a, b)$.

It is *unique up to associates*, and can be written as $d = xa + yb$ for some $x, y \in R$.

Proof

Existence. The ideal generated by a and b is

$$I = (a, b) = \{ua + vb : u, v \in R\}.$$

Since R is a PID, we can write $I = (d)$ for some $d \in I$, and so $d = xa + yb$.

Since $a, b \in (d)$, both $d \mid a$ and $d \mid b$ hold.

If c is a divisor of a & b, then $c \mid xa + yb = d$, so d is a GCD for a and b. ✓

Uniqueness. If d' is another GCD, then $d \mid d'$ and $d' \mid d$, so $d \sim d'$. ✓ □

Nice properties of PIDs

Corollary

If R is a PID, then every irreducible element is prime.

Proof

Let $p \in R$ be irreducible and suppose $p \mid ab$ for some $a, b \in R$.

If $p \nmid a$, then $\mathbf{gcd}(p, a) = 1$, so we may write $1 = xa + yp$ for some $x, y \in R$. Thus

$$b = (xa + yp)b = x(ab) + (yb)p\,.$$

Since $p \mid x(ab)$ and $p \mid (yb)p$, then $p \mid x(ab) + (yb)p = b$. $\square$

Not surprisingly, least common multiples also have a nice characterization in PIDs.

Proposition (HW)

If R is a PID, then any $a, b \in R^*$ have an LCM, $m = \mathbf{lcm}(a, b)$.

It is *unique up to associates*, and can be characterized as a generator of the ideal $I := (a) \cap (b)$.

Unique factorization domains

Definition

An integral domain is a unique factorization domain (UFD) if:

(i) Every nonzero element is a product of irreducible elements;

(ii) Every irreducible element is prime.

Examples

1. $\mathbb{Z}$ is a UFD: Every integer $n \in \mathbb{Z}$ can be uniquely factored as a product of irreducibles (primes):
$$n = p_1^{d_1} p_2^{d_2} \cdots p_k^{d_k}.$$
This is the *fundamental theorem of arithmetic*.

2. The ring $\mathbb{Z}[x]$ is a UFD, because every polynomial can be factored into irreducibles. But it is not a PID because the following ideal is not principal:
$$(2, x) = \{f(x) : \text{ the constant term is even}\}.$$

3. The ring R_{-5} is not a UFD because $9 = 3 \cdot 3 = (2 + \sqrt{-5})(2 - \sqrt{-5})$.

4. We've shown that (ii) holds for PIDs. Next, we will see that (i) holds as well.

Unique factorization domains

Theorem

If R is a PID, then R is a UFD.

Proof

We need to show Condition (i) holds: every element is a product of irreducibles. A ring is Noetherian if every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

stabilizes, meaning that $I_k = I_{k+1} = I_{k+2} = \cdots$ holds for some k.

Suppose R is a PID. It is not hard to show that R is Noetherian (HW). Define

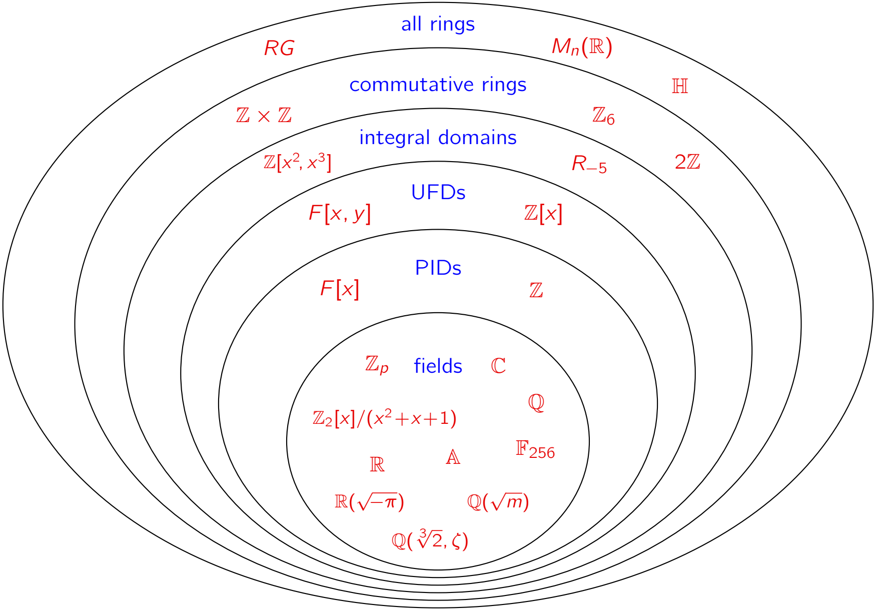
$$X = \{a \in R^* \setminus U(R) : a \text{ can't be written as a product of irreducibles}\}.$$

If $X \neq \emptyset$, then pick $a_1 \in X$. Factor this as $a_1 = a_2 b$, where $a_2 \in X$ and $b \notin U(R)$. Then $(a_1) \subsetneq (a_2) \subsetneq R$, and repeat this process. We get an ascending chain

$$(a_1) \subsetneq (a_2) \subsetneq (a_3) \subsetneq \cdots$$

that does not stabilize. This is impossible in a PID, so $X = \emptyset$. □

Summary of ring types



The Euclidean algorithm

Around 300 B.C., Euclid wrote his famous book, the *Elements*, in which he described what is now known as the Euclidean algorithm:

Proposition VII.2 (Euclid's *Elements*)

Given two numbers not prime to one another, to find their greatest common measure.

The algorithm works due to two key observations:

- If $a \mid b$, then $\gcd(a,b) = a$;
- If $a = bq + r$, then $\gcd(a,b) = \gcd(b,r)$.

This is best seen by an example: Let $a = 654$ and $b = 360$.

$$\begin{array}{ll}
654 = 360 \cdot 1 + 294 & \gcd(654,360) = \gcd(360,294) \\
360 = 294 \cdot 1 + 66 & \gcd(360,294) = \gcd(294,66) \\
294 = 66 \cdot 4 + 30 & \gcd(294,66) = \gcd(66,30) \\
66 = 30 \cdot 2 + 6 & \gcd(66,30) = \gcd(30,6) \\
30 = 6 \cdot 5 & \gcd(30,6) = 6.
\end{array}$$

We conclude that $\gcd(654,360) = 6$.

Euclidean domains

Loosely speaking, a Euclidean domain is a ring for which the Euclidean algorithm works.

Definition

An integral domain R is Euclidean if it has a degree function $d\colon R^* \to \mathbb{Z}$ satisfying:

(i) non-negativity: $d(r) \geq 0 \quad \forall r \in R^*$.

(ii) monotonicity: $d(a) \leq d(ab)$ for all $a, b \in R^*$.

(iii) division-with-remainder property: For all $a, b \in R$, $b \neq 0$, there are $q, r \in R$ such that

$$a = bq + r \qquad \text{with} \qquad r = 0 \quad \text{or} \quad d(r) < d(b)\,.$$

Note that Property (ii) could be restated to say: *If $a \mid b$, then $d(a) \leq d(b)$*;

Since 1 divides every $x \in R$,

$$d(1) \leq d(x), \qquad \text{for all } x \in R.$$

Similarly, if x divides 1, then $d(x) \leq d(1)$. Elements that divide 1 are the units of R.

Proposition

If u is a unit, then $d(u) = d(1)$. □

Euclidean domains

Examples

- $R = \mathbb{Z}$ is Euclidean. Define $d(r) = |r|$.
- $R = F[x]$ is Euclidean if F is a field. Define $d(f(x)) = \deg f(x)$.
- The Gaussian integers

$$R_{-1} = \mathbb{Z}[\sqrt{-1}] = \{a + bi \mid a, b \in \mathbb{Z}\}$$

  is Euclidean with degree function $d(a + bi) = a^2 + b^2$.

Proposition

If R is Euclidean, then $U(R) = \{x \in R^* \mid d(x) = d(1)\}$.

Proof

We've already established "$\subseteq$". For "$\supseteq$", Suppose $x \in R^*$ and $d(x) = d(1)$.

Write $1 = qx + r$ for some $q \in R$, and $r = 0$ or $d(r) < d(x) = d(1)$.

But $d(r) < d(1)$ is impossible, and so $r = 0$, which means $qx = 1$ and hence $x \in U(R)$. $\square$

Euclidean domains

Proposition

If R is Euclidean, then R is a PID.

Proof

Let $I \neq 0$ be an ideal and pick some $b \in I$ with $d(b)$ minimal.

Pick $a \in I$, and write $a = bq + r$ with either $r = 0$, or $d(r) < d(b)$.

This latter case is impossible: $r = a - bq \in I$, and by minimality, $d(b) \leq d(r)$.

Therefore, $r = 0$, which means $a = bq \in (b)$. Since a was arbitrary, $I = (b)$. □

Exercises.

(i) The ideal $I = (3, 2 + \sqrt{-5})$ is not principal in R_{-5}.

(ii) If R is an integral domain, then $I = (x, y)$ is not principal in $R[x, y]$.

Corollary

The rings R_{-5} (not a PID or UFD) and $R[x, y]$ (not a PID) are not Euclidean.

Algebraic integers

The algebraic integers are the roots of *monic* polynomials in $\mathbb{Z}[x]$. This is a subring of the algebraic numbers (roots of all polynomials in $\mathbb{Z}[x]$).

Assume $m \in \mathbb{Z}$ is square-free with $m \neq 0, 1$. Recall the quadratic field

$$\mathbb{Q}(\sqrt{m}) = \left\{ p + q\sqrt{m} \mid p, q \in \mathbb{Q} \right\}.$$

Definition

The ring R_m is the set of algebraic integers in $\mathbb{Q}(\sqrt{m})$, i.e., the subring consisting of those numbers that are roots of monic quadratic polynomials $x^2 + cx + d \in \mathbb{Z}[x]$.

Facts

- R_m is an integral domain with 1.
- Since m is square-free, $m \not\equiv 0 \pmod 4$. For the other three cases:

$$R_m = \begin{cases} \mathbb{Z}[\sqrt{m}] = \left\{ a + b\sqrt{m} : a, b \in \mathbb{Z} \right\} & m \equiv 2 \text{ or } 3 \pmod 4 \\ \mathbb{Z}\left[\frac{1+\sqrt{m}}{2}\right] = \left\{ a + b\left(\frac{1+\sqrt{m}}{2}\right) : a, b \in \mathbb{Z} \right\} & m \equiv 1 \pmod 4 \end{cases}$$

- R_{-1} is the Gaussian integers, which is a PID. (easy)
- R_{-19} is a PID. (hard)

Algebraic integers

Definition

For $x = r + s\sqrt{m} \in \mathbb{Q}(\sqrt{m})$, define the norm of x to be

$$N(x) = (r + s\sqrt{m})(r - s\sqrt{m}) = r^2 - ms^2 .$$

R_m is norm-Euclidean if it is a Euclidean domain with $d(x) = |N(x)|$.

Note that the norm is multiplicative: $N(xy) = N(x)N(y)$.

Exercises

Assume $m \in \mathbb{Z}$ is square-free, with $m \neq 0, 1$.

- $u \in U(R_m)$ iff $|N(u)| = 1$.
- If $m \geq 2$, then $U(R_m)$ is infinite.
- $U(R_{-1}) = \{\pm 1, \pm i\}$ and $U(R_{-3}) = \left\{ \pm 1, \pm \frac{1 \pm \sqrt{-3}}{2} \right\}$.
- If $m = -2$ or $m < -3$, then $U(R_m) = \{\pm 1\}$.

Euclidean domains and algebraic integers

Theorem

R_m is norm-Euclidean iff

$$m \in \{-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\}.$$

Theorem (D.A. Clark, 1994)

The ring R_{69} is a Euclidean domain that is *not* norm-Euclidean.

Let $\alpha = (1 + \sqrt{69})/2$ and $c > 25$ be an integer. Then the following degree function works for R_{69}, defined on the prime elements:

$$d(p) = \begin{cases} |N(p)| & \text{if } p \neq 10 + 3\alpha \\ c & \text{if } p = 10 + 3\alpha \end{cases}$$

Theorem

If $m < 0$ and $m \notin \{-11, -7, -3, -2, -1\}$, then R_m is not Euclidean.

Open problem

Classify which R_m's are PIDs, and which are Euclidean.

PIDs that are not Euclidean

Theorem

If $m < 0$, then R_m is a PID iff

$$m \in \{ \underbrace{-1, -2, -3, -7, -11}_{\text{Euclidean}}, -19, -43, -67, -163 \}.$$

Recall that R_m is norm-Euclidean iff

$$m \in \{-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\}.$$

Corollary

If $m < 0$, then R_m is a PID that is not Euclidean iff $m \in \{-19, -43, -67, -163\}$.

Algebraic integers

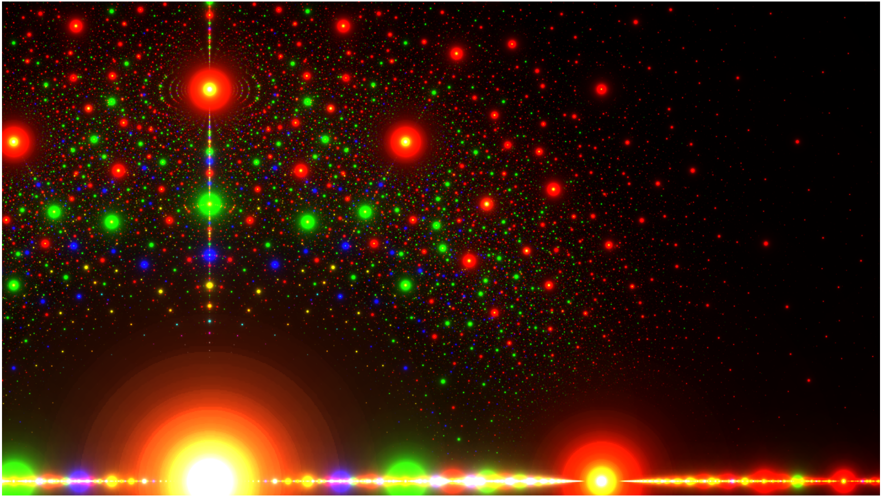


Figure: Algebraic numbers in the complex plane. Colors indicate the coefficient of the leading term: red $= 1$ (algebraic integer), green $= 2$, blue $= 3$, yellow $= 4$. Large dots mean fewer terms and smaller coefficients. Image from Wikipedia (made by Stephen J. Brooks).

Algebraic integers

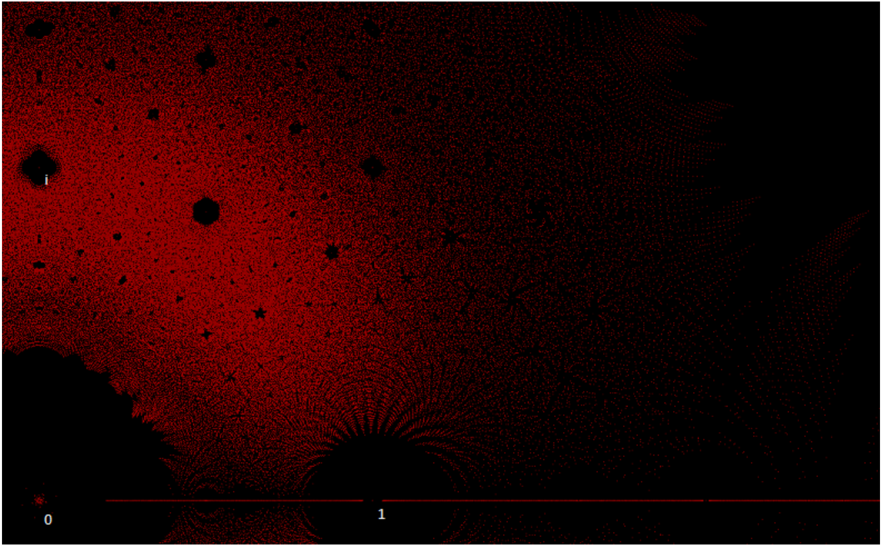
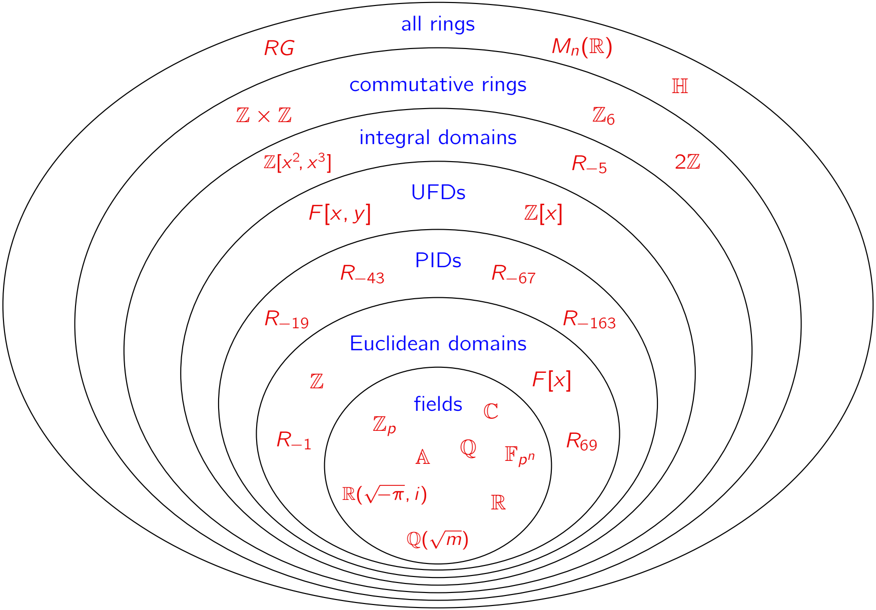


Figure: Algebraic integers in the complex plane. Each red dot is the root of a monic polynomial of degree ≤ 7 with coefficients from $\{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$. From Wikipedia.

Summary of ring types



Field of fractions

Rings allow us to add, subtract, and multiply, but not necessarily divide.

In any ring: if $a \in R$ is not a zero divisor, then $ax = ay$ implies $x = y$. *This holds even if a^{-1} doesn't exist.*

In other words, by allowing "divison" by non zero-divisors, we can think of R as a subring of a bigger ring that contains a^{-1}.

If $R = \mathbb{Z}$, then this construction yields the rational numbers, $\mathbb{Q}$.

If R is an integral domain, then this construction yields the field of fractions of R.

Goal

Given a commutative ring R, construct a larger ring in which $a \in R$ (that's not a zero divisor) has a multiplicative inverse.

Elements of this larger ring can be thought of as fractions. It will naturally contain an isomorphic copy of R as a subring:

$$R \hookrightarrow \left\{ \frac{r}{1} : r \in R \right\}.$$

From $\mathbb{Z}$ to $\mathbb{Q}$

Let's examine how one can construct the rationals from the integers.

There are many ways to write the same rational number, e.g., $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$

Equivalence of fractions

Given $a, b, c, d \in \mathbb{Z}$, with $b, d \neq 0$,

$$\frac{a}{b} = \frac{c}{d} \qquad \text{if and only if} \qquad ad = bc.$$

Addition and multiplication is defined as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

It is not hard to show that these operations are well-defined.

The integers $\mathbb{Z}$ can be identified with the subring $\left\{ \frac{a}{1} : a \in \mathbb{Z} \right\}$ of $\mathbb{Q}$, and every $a \neq 0$ has a multiplicative inverse in $\mathbb{Q}$.

We can do a similar construction in any commutative ring!

Rings of fractions

Blanket assumptions

- R is a commutative ring.
- $D \subseteq R$ is nonempty, multiplicatively closed $[d_1, d_2 \in D \Rightarrow d_1 d_2 \in D]$, and contains no zero divisors.
- Consider the following set of ordered pairs:

$$\mathcal{F} = \{(r, d) \mid r \in R,\ d \in D\},$$

Define an equivalence relation: $(r_1, d_1) \sim (r_2, d_2)$ iff $r_1 d_2 = r_2 d_1$. Denote this equvalence class containing (r_1, d_1) by $\frac{r_1}{d_1}$, or r_1/d_1.

Definition

The ring of fractions of D with respect to R is the set of equivalence classes, $R_D := \mathcal{F}/\sim$, where

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} := \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \qquad \text{and} \qquad \frac{r_1}{d_1} \times \frac{r_2}{d_2} := \frac{r_1 r_2}{d_1 d_2}.$$

Rings of fractions

Basic properties (HW)

1. These operations on $R_D = \mathcal{F}/\sim$ are well-defined.
2. $(R_D, +)$ is an abelian group with identity $\frac{0}{d}$, for any $d \in D$. The additive inverse of $\frac{a}{d}$ is $\frac{-a}{d}$.
3. Multiplication is associative, distributive, and commutative.
4. R_D has multiplicative identity $\frac{d}{d}$, for any $d \in D$.

Examples

1. Let $R = \mathbb{Z}$ (or $R = 2\mathbb{Z}$) and $D = R - \{0\}$. Then the ring of fractions is $R_D = \mathbb{Q}$.
2. If R is an integral domain and $D = R - \{0\}$, then R_D is a field, called the field of fractions.
3. If $R = F[x]$ and $D = \{x^n \mid n \in \mathbb{Z}\}$, then $R_D = F[x, x^{-1}]$, the Laurent polynomials over F.
4. If $R = \mathbb{Z}$ and $D = 5\mathbb{Z}$, then $R_D = \mathbb{Z}[\frac{1}{5}]$, which are "polynomials in $\frac{1}{5}$" over $\mathbb{Z}$.
5. If R is an integral domain and $D = \{d\}$, then $R_D = R[\frac{1}{d}]$, the set of all "polynomials in $\frac{1}{d}$" over R.

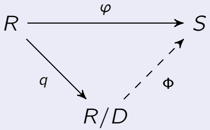
Universal property of the ring of fractions

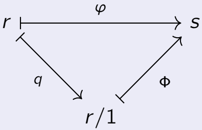
This says R_D is the "smallest" ring contaning R and all fractions of elements in D:

Theorem

Let S be any commutative ring with 1 and let $\varphi\colon R \hookrightarrow S$ be any ring embedding such that $\phi(d)$ is a unit in S for every $d \in D$.

Then there is a unique ring embedding $\Phi\colon R_D \to S$ such that $\Phi \circ q = \varphi$.





Proof

Define $\Phi\colon R_D \to S$ by $\Phi(r/d) = \varphi(r)\varphi(d)^{-1}$. This is well-defined and 1–1. (HW)

Uniqueness. Suppose $\Psi\colon R_D \to S$ is another embedding with $\Psi \circ q = \varphi$. Then

$$\Psi(r/d) = \Psi((r/1) \cdot (d/1)^{-1}) = \Psi(r/1) \cdot \Psi(d/1)^{-1} = \varphi(r)\varphi(d)^{-1} = \Phi(r/d).$$

Thus, $\Psi = \Phi$. $\square$