Topics: Linear maps, inner products, and orthogonality

1. Let $T\colon V \to W$ be a linear map between vector spaces. Prove that $\ker(T) := \{v \in V \mid T(v) = 0\}$ is a subspace of V.

2. Let $V = \mathcal{C}^1(\mathbb{R})$, the vector space of differentiable real-valued functions. Consider the linear operator $T = \frac{d}{dt} + 3$.

   (a) The kernel of T can be characterized precisely by the set of all funtions that solve a particular differential equation. Write down this equation.

   (b) Find the general solution for the differential equation you found in Part (a), and hence an explicit formula for $\ker(T)$.

   (c) Write down an explicit basis for the solution space, $\ker(T)$. What is the dimension of this vector space?

3. Let $\mathbf{v} = (3,4) \in \mathbb{R}^2$.

   (a) Compute $||\mathbf{v}|| := \sqrt{\mathbf{v}\cdot\mathbf{v}}$.

   (b) Recall that $\{\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)\}$ is an *orthonormal basis* for $\mathbb{R}^2$. Decompose $\mathbf{v}$ into this basis, i.e., write $\mathbf{v} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ for some $a_1, a_2 \in \mathbb{R}$.

   (c) Sketch $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{v}$ in $\mathbb{R}^2$. Graphically show what a_1 and a_2 represent in terms of the projection of $\mathbf{v}$ onto the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

   (d) The set $\{\mathbf{v}_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), \mathbf{v}_2 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\}$ is also an orthonormal basis. Decompose $\mathbf{v}$ into this basis, i.e., write $\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$ for some $b_1, b_2 \in \mathbb{R}$.

   (e) On a new set of axes, sketch $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}$ in $\mathbb{R}^2$. Graphically show what b_1 and b_2 represent in terms of the projection of $\mathbf{v}$ onto the unit vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

4. For this problem, consider the vector space $V = \mathbb{R}^3$ and use the vector dot product as the inner product.

   (a) Show that the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

   $$\mathbf{v}_1 = (1,2,-2)\,, \qquad \mathbf{v}_2 = (0,1,1)\,, \qquad \mathbf{v}_3 = (-4,1,-1)\,.$$

   is an orthogonal set, but not orthonormal.

   (b) Normalize $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ to get an *orthonormal* basis of $\mathbb{R}^3$. That is, compute the following:

   $$\mathcal{B} = \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\} \qquad \text{where} \quad \mathbf{n}_i = \frac{\mathbf{v}_i}{||\mathbf{v}_i||}\,.$$

   (c) Use the dot product to express the vector $\mathbf{w} = (1,2,3)$ in terms of $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$. That is, find C_1, C_2, and C_3 such that

   $$\mathbf{w} = C_1\mathbf{n}_1 + C_2\mathbf{n}_2 + C_3\mathbf{n}_3\,.$$

5. Let $\mathbb{R}_3[x] = \{a_3x^3 + a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R}\}$, the vector space of polynomials of degree at most 3. Define the following *inner product* on $\mathbb{R}_3[x]$:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx\,.$$

(a) Verify that this is indeed an inner product on $\mathbb{R}_3[x]$.

(b) Consider the two sets

$$\mathcal{B}_1 = \{1, x, 3x^2 - 1, 5x^3 - 3x\}\,, \qquad \mathcal{B}_2 = \{1, x, x^2, x^3\}$$

that are both bases for $\mathbb{R}_3[x]$. Show that $\mathcal{B}_1$ is an orthogonal set, but $\mathcal{B}_2$ is not. (The set $\mathcal{B}_1$ are the first four *Legendre polynomials*, $P_n(x)$ for $n = 0, \dots, 3$. When we study Sturm-Liouville theory, we will see why the Legendre polynomials are always orthogonal!)

(c) For each $f \in \mathcal{B}_1$, compute the *norm* of f, which is defined as $||f|| = \langle f, f \rangle^{1/2}$. Find an *orthonormal* basis for $\mathbb{R}_3[x]$ by normalizing the elements in $\mathcal{B}_1$.

(d) Consider the polynomial $f(x) = 3x^3 - 2x^2 + 4$. Use orthogonality to write $f(x)$ using the elements in $\mathcal{B}_1$. That is, find C_0, C_1, C_2, and C_3 such that

$$3x^3 - 2x^2 + 4 = C_0 + C_1x + C_2(3x^2 - 1) + C_3(5x^3 - 3x).$$