

## TOPICS: THE FROBENIUS METHOD AND BESSEL'S EQUATION

1. The differential equation  $(1 - x^2)y'' - xy' + p^2y = 0$ , where  $p$  is a constant, is known as *Chebyshev's equation*. It can be rewritten in the form

$$y'' + P(x)y' + Q(x)y = 0, \quad P(x) = -\frac{x}{1-x^2}, \quad Q(x) = \frac{p^2}{1-x^2}.$$

- (a) If  $P(x)$  and  $Q(x)$  are represented as a power series about  $x_0 = 0$ , what is the radius of convergence of these power series?
- (b) Assume that the general solution has the form  $\sum_{n=0}^{\infty} a_n x^n$ , and find a recurrence for  $a_{n+2}$  in terms of  $a_n$ . [*Hint*: Before plugging back in, multiply through by  $1 - x^2$ .]
- (c) Use the recurrence to determine  $a_n$  in terms of  $a_0$  and  $a_1$ , for  $2 \leq n \leq 9$ .
- (d) For each  $p \in \mathbb{N}$ , there is a unique polynomial solution  $T_p(x)$  known as the *Chebyshev polynomial* of degree  $p$ . Find  $T_3(x)$ .
2. For each of the following ODEs, determine whether  $x = 0$  is an ordinary or singular point. If it is singular, determine whether it is regular or not. (Remember, first write each ODE in the form  $y'' + P(x)y' + Q(x)y = 0$ .)

(a)  $y'' + xy' + (1 - x^2)y = 0$

(c)  $x^2y'' + 2xy' + (\cos x)y = 0$ .

(b)  $y'' + (1/x)y' + (1 - (1/x^2))y = 0$ .

(d)  $x^3y'' + 2xy' + (\cos x)y = 0$ .

3. Consider the differential equation  $3xy'' + y' + y = 0$ . Since  $x_0 = 0$  is a regular singular point, there is a generalized power series solution of the form  $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ .

- (a) Determine the indicial equation (solve for  $r$ ) and the recurrence relation for the coefficients.
- (b) Find two linearly independent generalized power series solutions (i.e., a *basis* for the solution space).
- (c) Determine the radius of convergence of each of these solutions. [*Hint*: First compute the radius of convergence of  $xP(x)$  and  $x^2Q(x)$  and apply Frobenius].
4. Consider the differential equation  $2xy'' + y' + xy = 0$ . Since  $x_0 = 0$  is a regular singular point, there is a solution of the form  $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ .
- (a) Determine the indicial equation (solve for  $r$ ) and the recursion formula.
- (b) Find a basis for the solution space and use this to write the general solution.
- (c) What is the radius of convergence of each of these two linearly independent solutions?

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5. Consider the differential equation  $xy'' + 2y' - xy = 0$ .
- (a) Show that  $x = 0$  is a regular singular point.
  - (b) Show that if  $a_0 = 0$ , then  $r = -1$  is one solution for the indicial equation.
  - (c) For  $r = -1$  and  $a_0 = 0$ , find the recurrence relation for  $a_{n+2}$  in terms of  $a_n$ .
  - (d) Still assuming that  $a_0 = 0$ , write the solution from (b) as a generalized power series.
  - (e) Write this solution using a standard hyperbolic trigonometric function.
6. Consider the ODE  $y'' + e^{-x}y = 0$ . Change variables by setting  $t = 2e^{-x/2}$ , which will reduce this ODE to a Bessel's equation. Using the known solution to Bessel's equation, back-substitute to determine the solution  $y(x)$  to the original ODE.