Lecture 3.1: Fourier series and orthogonality

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Some history

Ancient mathematicians dogmatically believed that only positive whole numbers could exist.

However, using basic arithmetic, they could create the negative numbers and the rational numbers (fractions).

Obviously, we know now of the existence of irrational numbers. These cannot be expressed as fractions, but there are fractions that are "arbitrarily close" to them.

Specifically, this means that they arise as limits of sequences of rational numbers.

For example, the number π is the limit of the following sequence.

$$\begin{aligned} x_0 &= 3 \\ x_1 &= 3.1 \\ x_2 &= 3.14 \\ x_3 &= 3.141 \\ x_4 &= 3.1415 \\ x_5 &= 3.14159 \\ &\vdots \end{aligned}$$

As we know from calculus, an alternative way to express this is as a series (sequence of partial sums):

$$\pi = 3 + .1 + .04 + .001 + .0005 + .00009 + \cdots$$

Some history

Many students have a similar epiphany as the ancient Greeks when they learn about Taylor series in calculus.

For example, the function $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is the limit of the following sequence

$$\begin{aligned}
f_0(x) &= 1 \\
f_1(x) &= 1 + x \\
f_2(x) &= 1 + x + \tfrac{1}{2}x^2 \\
f_3(x) &= 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 \\
f_4(x) &= 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 \\
f_4(x) &= 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \tfrac{1}{120}x^5 \\
&\vdots
\end{aligned}$$

Big idea

Even though functions like e^x don't technically exist in the vector space of polynomials, we can, for all intents and purposes, treat them like they do.

That said, we have to be careful regarding convergence. For example, the following formal power series is *not* a real-valued function $\mathbb{R} \to \mathbb{R}$:

$$g(x) = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Introduction

Recall the definition of a vector space: a set V (of vectors) and a set $\mathbb{F}$ (of scalars) that is

- Closed under addition: $v, w \in V \implies v + w \in V$,
- Closed under scalar multiplication: $v \in V$, $c \in \mathbb{F} \implies cv \in V$.

The infinite-dimensional space $\mathbb{R}[x]$ of polynomials has basis $\{x^k \mid k = 0, 1, 2, \dots\}$. That is,

$$\mathbb{R}[x] = \text{Span}\{1, x, x^2, x^3, \dots\}.$$

Consider the vector space spanned by the following set of sine and cosine waves:

$$V = \text{Span}\Big\{\{1,\ \cos x,\ \cos 2x, \dots\} \cup \{\sin x,\ \sin 2x, \dots\}\Big\}.$$

Think of these elements as smooth "sound waves."

Just like how many functions such as e^x are "arbitrarily close" to polynomials, there are many 2π-periodic functions that are "arbitrarily close" to elements of V.

Examples include square waves, triangle waves, and much more.

Technically, we say that the vector space V is dense in the set $\text{Per}_{2\pi}(\mathbb{R})$.

Like we did with e^x and $\mathbb{R}[x]$, we can for all intents and purposes, treat the set $\text{Per}_{2\pi}(\mathbb{R})$ as a vector space with basis $\{1, \cos nx, \sin nx : n \in \mathbb{N}\}$ and allow infinite sums.

Inner products

Back to $\mathbb{R}^n$

Recall that once we defined an inner product on $\mathbb{R}^n$, we were able to:

- measure the lengths of vectors; $||\mathbf{v}|| := \sqrt{\mathbf{v}\cdot\mathbf{v}}$,
- measure the angles between vectors; $\measuredangle(\mathbf{v},\mathbf{w}) := \cos^{-1}\big(\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||}\big)$,
- project vectors onto unit vectors: $\operatorname{Proj}_{\mathbf{n}}\mathbf{v} := (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}$.
- decompose a vector $\mathbf{v}\in\mathbb{R}^n$ into components using an orthonormal basis:

$$\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n = (a_1,\dots,a_n) \qquad \text{where } a_i = \operatorname{proj}_{\mathbf{e}_i}(\mathbf{v}) = \mathbf{v}\cdot\mathbf{e}_i.$$

Definition

The inner product ("generalized dot product") on $\mathrm{Per}_{2\pi}(\mathbb{R})$ is defined to be:

$$\langle f, g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx\,.$$

Proposition

With respect to this inner product, the set

$$\mathcal{B}_{2\pi} = \left\{ \begin{array}{lllll} \frac{1}{\sqrt{2}}, & \cos x, & \cos 2x, & \cos 3x, & \dots \\ & \sin x, & \sin 2x, & \sin 3x, & \dots \end{array} \right\}$$

is an orthonormal basis for $\mathrm{Per}_{2\pi}(\mathbb{R})$!

Note that this just means that the following (easily verifiable) formulas hold:

$$\big\langle \cos nx, \cos mx \big\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx\, \cos mx\, dx = \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

$$\big\langle \sin nx, \sin mx \big\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx\, \sin mx\, dx = \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

$$\big\langle \cos nx, \sin mx \big\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx\, \sin mx\, dx = 0.$$

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is an orthonormal basis for $\mathrm{Per}_{2\pi}(\mathbb{R})$!

The utility of having an inner product

Now that we have an inner product on $\mathrm{Per}_{2\pi}(\mathbb{R})$ and an orthonormal basis, we can

- project vectors onto unit vectors: $\mathrm{proj}_{u(x)}\, f(x) := \langle f, u \rangle$.
- decompose a vector $f \in \mathrm{Per}_{2\pi}(\mathbb{R})$ into components using our orthonormal basis $\mathcal{B}_{2\pi}$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \qquad \text{where} \quad \begin{aligned} a_n &= \mathrm{proj}_{\cos nx}(f) = \langle f, \cos nx \rangle \\ b_n &= \mathrm{proj}_{\sin nx}(f) = \langle f, \sin nx \rangle. \end{aligned}$$

Fourier series

Definition / Theorem

Let f be a piecewise continuous 2π-periodic function. The Fourier series of f is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

and the formulas for the Fourier coefficients are given by

$$a_n = \mathrm{proj}_{\cos nx}(f) = \langle f, \cos nx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \mathrm{proj}_{\sin nx}(f) = \langle f, \sin nx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Remarks

- These formulas hold for all n, including $n = 0$.
- Even though the vector space spanned by $\{1, \cos nx, \sin nx \mid n \in \mathbb{N}\}$ technically only consists of finite sums of sines and cosines, if one allows infinite series, this basically works for piecewise continuous functions as well.
- At times, it may be easier to integrate over $[0, 2\pi]$ rather than $[-\pi, \pi]$.

Fourier series

Let $\text{Per}_{2L}(\mathbb{R})$ be the vector space of all real-valued $2L$-periodic functions

$$V = \text{Span}\left\{\{1,\, \cos\tfrac{\pi x}{L},\, \cos\tfrac{2\pi x}{L}, \dots\} \cup \{\sin\tfrac{\pi x}{L},\, \sin\tfrac{2\pi x}{L}, \dots\}\right\}.$$

Define an inner product on $\text{Per}_{2L}(\mathbb{R})$ by

$$\langle f, g\rangle := \frac{1}{L}\int_{-L}^{L} f(x)g(x)\,dx\,.$$

Definition / Theorem

Let f be a piecewise continuous $2L$-periodic function. The Fourier series of f is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\tfrac{\pi n x}{L}) + b_n \sin(\tfrac{\pi n x}{L}),$$

and the formulas for the Fourier coefficients are given by

$$a_n = \text{proj}_{\cos(n\pi x/L)}(f) = \left\langle f,\, \cos\tfrac{n\pi x}{L}\right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\cos\tfrac{n\pi x}{L}\,dx$$

$$b_n = \text{proj}_{\sin(n\pi x/L)}(f) = \left\langle f,\, \sin\tfrac{n\pi x}{L}\right\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\sin\tfrac{n\pi x}{L}\,dx.$$