Lecture 4.5: Generalized Fourier series

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

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Last time

Definition

A Sturm-Liouville equation is a 2nd order ODE of the following form:

$$-(p(x)y')' + q(x)y = \lambda w(x)y, \qquad \text{where } p(x),\ q(x),\ w(x) > 0.$$

We are usually interested in solutions $y(x)$ on a bounded interval $[a, b]$, under some homogeneous BCs:

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \qquad \alpha_1^2 + \alpha_2^2 > 0$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0 \qquad \beta_1^2 + \beta_2^2 > 0.$$

Together, this BVP is called a Sturm-Liouville (SL) problem.

Main theorem

Given a Sturm-Liouville problem:

(a) The eigenvalues are real and can be ordered so $\lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty$.

(b) Each eigenvalue λ_i has a unique (up to scalars) eigenfunction $y_i(x)$.

(c) W.r.t. the inner product $\langle f, g \rangle := \int_a^b f(x)\overline{g(x)}w(x)\,dx$, the eigenfunctions form an orthonormal basis on the subspace of functions $\mathcal{C}^\infty_{\alpha,\beta}[a, b]$ that satisfy the BCs.

What this means

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Definition

If $f \in \mathcal{C}^\infty_{\alpha,\beta}[a,b]$, then f can be written uniquely as a linear combination of the eigenfunctions. That is,

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \qquad \text{where } c_n = \frac{\langle f, y_n \rangle}{\langle y_n, y_n \rangle} = \frac{\int_a^b f(x)\overline{y_n(x)}w(x)\,dx}{\int_a^b ||y_n(x)||^2 w(x)\,dx}.$$

This is called a generalized Fourier series with respect to the orthogonal basis $\{y_n(x)\}$ and weighting function $w(x)$.

Example 1 (Dirichlet BCs)

$-y'' = \lambda y, \quad y(0) = 0, \quad y(\pi) = 0$ is an SL problem with:

- Eigenvalues: $\lambda_n = n^2, \quad n = 1, 2, 3, \ldots$.
- Eigenfunctions: $y_n(x) = \sin(nx)$.

The orthogonality of the eigenvectors means that

$$\langle y_m,\, y_n\rangle := \int_0^\pi y_m(x) y_n(x) w(x)\, dx = \int_0^\pi \sin(mx)\,\sin(nx)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi/2 & \text{if } m = n. \end{cases}$$

Note that this means that $||y_n|| := \langle y_n,\, y_n\rangle^{1/2} = \sqrt{\pi/2}$.

Fourier series: *any* function $f(x)$, continuous on $[0,\pi]$ satisfying $f(0) = 0, \; f(\pi) = 0$ can be written *uniquely* as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{\langle f,\, \sin nx\rangle}{\langle \sin nx,\, \sin nx\rangle} = \frac{\int_0^\pi f(x) \sin nx\, dx}{||\sin nx||^2} = \frac{2}{\pi} \int_0^\pi f(x) \sin nx\, dx.$$

Example 2 (Neumann BCs)

$-y'' = \lambda y, \quad y'(0) = 0, \quad y'(\pi) = 0$ is an SL problem with:

- Eigenvalues: $\lambda_n = n^2, \quad n = 0, 1, 2, 3, \dots$.
- Eigenfunctions: $y_n(x) = \cos(nx)$.

The orthogonality of the eigenvectors means that

$$\langle y_m,\, y_n\rangle := \int_0^\pi y_m(x) y_n(x) w(x)\,dx = \int_0^\pi \cos(mx)\,\cos(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi/2 & \text{if } m = n > 0. \end{cases}$$

Note that this means that $||y_n|| := \langle y_n,\, y_n\rangle^{1/2} = \begin{cases} \sqrt{\pi/2} & n > 0 \\ \sqrt{\pi} & n = 0. \end{cases}$

Fourier series: *any* function $f(x)$, continuous on $[0, \pi]$ satisfying $f'(0) = 0, \;\; f'(\pi) = 0$ can be written *uniquely* as

$$f(x) = \sum_{n=0}^{\infty} a_n \cos nx$$

where

$$a_n = \frac{\langle f,\, \cos nx\rangle}{\langle \cos nx,\, \cos nx\rangle} = \frac{\int_0^\pi f(x) \cos nx\,dx}{||\cos nx||^2} = \frac{2}{\pi}\int_0^\pi f(x) \cos nx\,dx.$$

The same formula holds for a_0 if you let the $n = 0$ (constant) term be $\frac{a_0}{2}$ rather than a_0.

Example 3 (Mixed BCs)

$-y'' = \lambda y, \quad y(0) = 0, \quad y'(\pi) = 0$ is an SL problem with:

- Eigenvalues: $\lambda_n = \left(n + \frac{1}{2}\right)^2, \quad n = 0, 1, 2, \dots$.
- Eigenfunctions: $y_n(x) = \sin\left(n + \frac{1}{2}\right)x$.

The orthogonality of the eigenvectors means that

$$\langle y_m, y_n \rangle := \int_0^\pi \sin\left(m + \tfrac{1}{2}\right)x \, \sin\left(n + \tfrac{1}{2}\right)x \, w(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi/2 & \text{if } m = n. \end{cases}$$

Note that this means that $||y_n|| := \langle y_n, y_n \rangle^{1/2} = \sqrt{\pi/2}$.

(Generalized?) Fourier series: *any* function $f(x)$, continuous on $[0, \pi]$ satisfying $f(0) = 0$, $f'(\pi) = 0$ can be written *uniquely* as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(n + \tfrac{1}{2}\right)x$$

where

$$b_n = \frac{\langle f, \sin\left(n + \frac{1}{2}\right)x \rangle}{\langle \sin\left(n + \frac{1}{2}\right)x, \sin\left(n + \frac{1}{2}\right)x \rangle} = \frac{\int_0^\pi f(x) \sin\left(n + \frac{1}{2}\right)x \, dx}{|| \sin\left(n + \frac{1}{2}\right)x||^2} = \frac{2}{\pi} \int_0^\pi f(x) \sin\left(n + \tfrac{1}{2}\right)x \, dx.$$

Example 4 (Robin BCs)

$-y'' = \lambda y, \quad y(0) = 0, \quad y(1) + y'(1) = 0$ is an SL problem with:

- Eigenvalues: $\lambda_n = \omega_n^2, \quad n = 1, 2, 3, \dots$ [ω_n's are the positive roots of $y(x) = x - \tan x$].
- Eigenfunctions: $y_n(x) = \sin(\omega_n x)$.

The orthogonality of the eigenvectors means that

$$\langle y_m, \, y_n \rangle := \int_0^1 y_m(x) y_n(x) w(x)\, dx = \int_0^1 \sin(\omega_m x) \, \sin(\omega_n x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ ??? & \text{if } m = n. \end{cases}$$

Though there isn't a nice closed-form solution, we still have $||y_n|| := \langle y_n, \, y_n \rangle^{1/2}$.

Generalized Fourier series: *any* function $f(x)$, continuous on $[0, 1]$ satisfying $f(0) = 0$, $f(1) + f'(1) = 0$ can be written *uniquely* as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \omega_n x$$

where

$$b_n = \frac{\langle f, \, \sin \omega_n x \rangle}{\langle \sin \omega_n x, \, \sin \omega_n x \rangle} = \frac{\int_0^1 f(x) \sin \omega_n x \, dx}{|| \sin \omega_n x||^2} = \frac{\int_0^1 f(x) \sin \omega_n x \, dx}{\int_0^1 (\sin \omega_n x)^2 \, dx}.$$