Lecture 6.3: Solving PDEs with Laplace transforms

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Introduction

A function $f\colon [0,\infty) \to \mathbb{C}$ has *exponential order*, if $|f(t)| \leq ce^{at}$ holds for sufficiently large t, where $a, c > 0$.

Definition

Let $f\colon [0,\infty) \to \mathbb{R}$ be a piecewise continuous function of exponential order. The Laplace transform of f is

$$(\mathcal{L}f)(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

Property	time-domain	frequency domain
Linearity	$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
Time / phase-shift	$f(t-c)$	$e^{-cs}F(s)$
Multiplication by exponential	$e^{ct}f(t)$	$F(s-c)$
Dilation by $c > 0$	$f(ct)$	$\frac{1}{c}F(s/c)$
Differentiation	$\dfrac{df(t)}{dt}$	$sF(s) - f(0)$
Multiplication by t	$tf(t)$	$-F'(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s) \cdot F_2(s)$
	$f_1(t) \cdot f_2(t)$	$(F_1 * F_2)(s)$

Solving an ODE with a Laplace transform

Example 1

Solve the initial value problem

$$u'' + u = 0, \qquad u(0) = 0, \qquad u'(0) = 1.$$

Laplace transform of a multivariate function

Definition

For a function $u(x,t)$ of two variables, define its Laplace transform by

$$(\mathcal{L}u)(x,s) = U(x,s) = \int_0^\infty u(x,t)e^{-st}\,dt.$$

Remark

The Laplace transform turns t-derivatives into multiplication, and leaves x-derivatives unchanged:

- $(\mathcal{L}u_x)(x,s) = U_x(x,s)$,
- $(\mathcal{L}u_{xx})(x,s) = U_{xx}(x,s)$,
- $(\mathcal{L}u_t)(x,s) = sU(x,s) - u(x,0)$,
- $(\mathcal{L}u_{tt})(x,s) = s^2U(x,s) - su(x,0) - u_x(x,0)$.

Convolution

Definition

The convolution of $f(t)$ and $g(t)$ is the function $(f * g)(t) := \int_0^t f(u)g(t-u)\,du$.

Properties

- $f * g = g * f$,
- $f * (g * h) = (f * g) * h$.

Convolution theorem

For functions f and g,

$$\mathcal{L}(f * g)(s) = F(s)G(s).$$

By taking the inverse Fourier transform of both sides, it follows that

$$(f * g)(t) = \mathcal{L}^{-1}\big(F(s)G(s)\big).$$

Solving a PDE with a Laplace transform

Example 2: the diffusion equation on a semi-infinite domain

Let $u(x,t)$ be the concentration of a chemical dissolved in a fluid, where $x > 0$. Consider the following B/IVP problem for the diffusion equation:

$$u_t = u_{xx}, \qquad u(0,t) = 1, \qquad u(x,0) = 0, \qquad u(x,t) \text{ bounded.}$$

Solving a PDE with a Laplace transform

Example 3: the diffusion equation on a semi-infinite domain

Let $u(x,t)$ be the concentration of a chemical dissolved in a fluid, where $x > 0$. Consider the following B/IVP problem for the diffusion equation:

$$u_t = u_{xx}, \qquad u(0,t) = f(t), \qquad u(x,0) = 0, \qquad u(x,t) \text{ bounded.}$$