1. In our first model of air resistance, the resistance force depends only on the velocity. This leads to the differential equation $mv' = -mg - rv$, where $-mg$ is the gravitation force and $R(v) = -rv$ is the air resistance force. For an object that drops a considerable distance, such as a parachutist, there is a dependence on the altitude as well. It is reasonable to assume that the resistance force is proportional to air pressure, as well as velocity. Furthermore, to a first-order approximation, the air pressure varies exponentially with the altitude (i.e., it is proportional to e^{-ax}, where a is a constant and x is the altitude). Propose and justify (but do not solve) a differential equation model (in x instead of v) for the velocity of a falling object subject to such a resistance force. Recall that $x' = v$.

2. The population of snakes on a plane is believed to be growing according to the logistic equation:

$$y' = ry\,(1 - y/M) \qquad\qquad \text{Solution } y(t) = \frac{M}{1 + Ce^{-rt}}\,.$$

The maximum number of snakes that can live on the plane is 1000. Initially, the population is 500, and at this time, the rate of increase of snakes is 100 per month.

   (a) How many months until the population reaches 90% of the maximum?

   (b) Sketch this solution curve in the (t, y)-plane, and the two steady-state solutions.

3. Let $T(t)$ be the temperature of a cup of water at time t, in hours. Newton's law of cooling says that at any t, the rate of change $T'(t)$ is proportional to the difference in ambient temperature and $T(t)$, which can be described by the differential equation

$$T' = k(A - T).$$

The ambient temperature A need not be constant; suppose it varies sinusoidally with time with a period of 24 hours. At 6am, the ambient temperature is at its minimum of 40° and at 6pm, its maximum of 60°.

   (a) Write down a differential equation (that is, find $A(t)$) that models the temperature of the cup of water. Let $t = 0$ be noon.

   (b) Give a physical *and* mathematical explanation why there is no steady-state solution.

   (c) The long-term behavior of a solution to this equation is a sinusoid; explain why. What do you expect the period, amplitude, and phase shift of this solution to be (qualitatively) compared to $A(t)$?

   (d) Without actually solving anything, sketch a graph of several solutions to this equation corresponding to different initial temperatures.

   (e) Suppose that instead of measuring the temperature of a cup of water, $T(t)$ measures the temperature of a pond. What parameters would this change in the differential equation? What would your graphs in Part (d) look like? Sketch solutions corresponding to the same initial conditions.