1. In this problem, we will investigate 2-cycles in the logistic map $f(x) = rx(1-x)$.

   (a) The presence of a two cycle $p, q, p, q, \dots$ means that p is a fixed point of $f^2(x) := f(f(x))$, and hence a root of the polynomial $f^2(x) - x = 0$.

   (b) The fixed points of the logistic map, $x^* = 0$ and $x^* = \frac{r-1}{r}$, are both roots of this polynomial. Therefore,
   $$f^2(x) = x\big(x - \tfrac{r-1}{r}\big)g(x).$$
   Find $g(x)$ (feel free to use a computer) and its roots.

   (c) Determine for what values of r give rise to a 2-cycle, with justification.

   (d) The two cycle $p, q, p, q, \dots$ of f is *stable* if p and q are stable fixed points for f^2. Compute
   $$\lambda := \frac{d}{dx}(f^2(x))_{x=p}.$$
   Then substitute r back in, and determine when $|\lambda| < 1$. This gives the values of r for which the 2-cycle is stable.

2. Consider the following instance of the discrete logistic equation:
   $$P_{t+1} = P_t(1 + r(1 - P_t/M))$$
   Find the two equilibirum points, P^*. Use the technique of *linearization* to find the stability of these points. That is, plug $P_t \approx P^* + p_t$ and $P_{t+1} \approx P^* + p_{t+1}$ into the difference equation and express the perturbation p_{t+1} in terms of p_t, disregarding the non-linear terms.

3. The discrete logistic and the Ricker population models when written as $P_{t+1} = F(P_t)$ have the property that for small values of P_t, the graph of $F(x)$ lies *above* the line $y = x$ This means that $F(P_t) > P_t$ for small value of P_t. Consider a model for which $F(P_t) < P_t$ for small values of P_t. Explain the affect of this feature on population dynamics. Why might this be a biologically important feature? (The resulting behavior is sometimes known as an *Allee effect.*)

4. Construct a simple model showing an Allee effect as follows.

   (a) Explain why for some $0 < L < K$, the per-capita growth should be
   $$\frac{\Delta P}{P} < 0, \quad \text{when } 0 < P < L \ \text{ or } P > K\,,$$
   $$\frac{\Delta P}{P} > 0, \quad \text{when } L < P < K\,.$$
   Sketch a possible graph of $\Delta P/P$ vs. P.

   (b) Explain why $\Delta P/P = P(K-P)(P-L)$ has the qualitative features desired.

   (c) Investigate the resulting model using the MATLAB programs `onepop` and `cobweb` (available on the Math 4500 webpage) for some choices of K and L. Print out, or sketch the results of a few sample trials. Is the behavior as expected?

   (d) What features of this modeling equation are unrealistic? How might the model be improved?