# (3) Analyzing non-linear models

<u>Transient</u> behavior: (initial, few iterations). Will usually "die out."

<u>Long-term behavior</u>: Often independent, or almost independent, of IC?

- $P_{t+1} = P_t$ is a <u>fixed point</u> (or equilibrium, or steady-state).
- We can also have longer <u>periodic cycles</u>:

$$\underbrace{P_t,\ P_{t+1},\ \ldots,}_{\text{cycle.}}\ \underbrace{P_{t+k}}_{=P_t},\ \underbrace{P_{t+(k+1)}}_{=P_{t+1}},\ \underbrace{P_{t+(k+2)}}_{=P_{t+2}},\ \ldots$$

Note that fixed points are cycles of length 1.

How to find fixed points.

2 ways: (i) Set $\Delta P = 0$, solve for $P$.

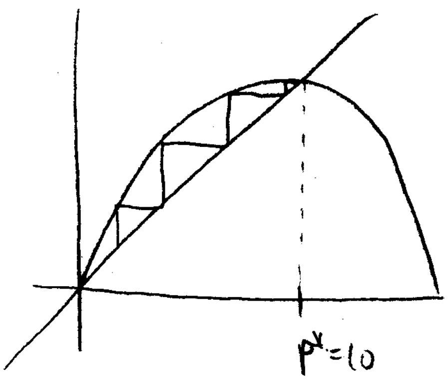
(ii) Set $P_t = P_{t+1} = P^*$, solve for $P^*$

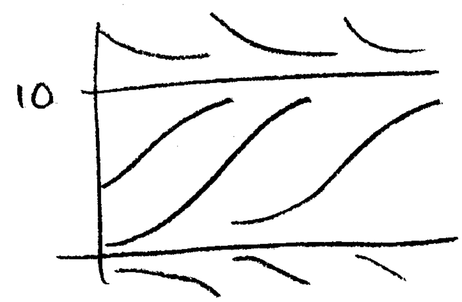
<u>Ex</u>: $P_{t+1} = P_t\left(1 + .7\left(1 - \frac{P_t}{10}\right)\right)$

Solve $P^* = P^*\left(1 - .7\left(1 - \frac{P^*}{10}\right)\right) \Rightarrow P^* = 0$ or $P^* = 10$.

Graphically: $P_{t+1} = F(P_t) = P_t\left(1 + .7\left(1 - \frac{P_t}{10}\right)\right)$

$F(x) = x(1.7 - .07x)$




Equilibria can be stable or unstable

"attracting" e.g., $p^* = 10$ "repelling", e.g., $p^* = 0$.

## Linearization

Key idea: Stability depends on behavior close to equilibrium.

Consider $P_t \approx P^*$, say $P_t = P^* + p_t$ small "perturbation"

$P_{t+1} = P^* + p_{t+1}$ Find this. Is it growing or shrinking?

Ex: $P_{t+1} = P_t\left(1 + .7\left(1 - \frac{P_t}{10}\right)\right)$, $P^* = 0, 10$

Analyze $P^* = 10$: Substitute $\begin{cases} P_t = 10 + p_t \\ P_{t+1} = 10 + p_{t+1} \end{cases}$ into eq'n.

$$10 + p_{t+1} = (10 + p_t)\left(1 + .7\left(1 - \frac{10 + p_t}{10}\right)\right)$$

$$10 + p_{t+1} = (10 + p_t)(1 - .07 p_t)$$

$$\cancel{10} + p_{t+1} = \cancel{10} + .3p_t - .07 p_t^2$$

$$\boxed{p_{t+1} = .3p_t - .07p_t^2} \approx .3p_t \text{ for small } p_t.$$

$\Rightarrow$ Perturbations shrink $\Rightarrow$ $P^* = 10$ is stable.

Exercise: Check $P^* = 0$ is unstable

Remark: This is the "discrete" version of the derivative:

$$\frac{p_{t+1}}{p_t} = \frac{P_{t+1} - P^*}{P_t - P^*} = \underbrace{\frac{F(P_t) - F(P^*)}{P_t - P^*}}$$

$\lim_{P_t \to P^*}$ is the derivative, $F'(P^*)$.

Theorem: If $P_{t+1} = F(P_t)$ has an equilibrium $P^*$, then

$|F'(P^*)| > 1 \Rightarrow P^*$ unstable

$|F'(P^*)| < 1 \Rightarrow P^*$ stable

$|F'(P^*)| = 1 \Rightarrow$ ? (inconclusive).

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# Bifurcation

Idea: Understand the dynamics qualitatively as parameter $r$ changes.

Ex: Logistic model: $\Delta P = rP(1-\frac{P}{K})$ , $r>0$

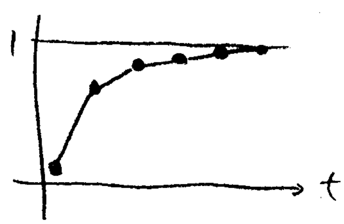
Normalize $K=1$ : $\boxed{\Delta P = rP(1-P)}$ $\quad F(x) = x(1-x)$

Fixed points: $P^* = 0$ $\quad p_{t+1} \approx (1+r)p_t$ $\quad$ unstable; $1+r>1$

$P^* = 1$ $\quad p_{t+1} \approx (1-r)\, p_t$ $\quad$ ???

↑ stretching factor

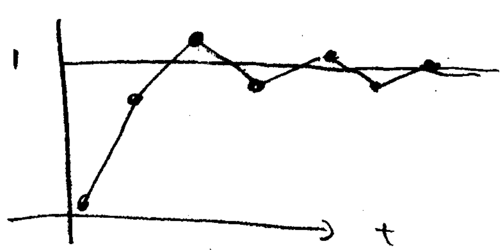
Case 1: $0<r<1 \Rightarrow 0<1-r<1 \Rightarrow p_{t+1} = \underbrace{(1-r)}_{>0} p_t$



$\Rightarrow p_t \to 0$, doesn't change sign

This is "overdamped"

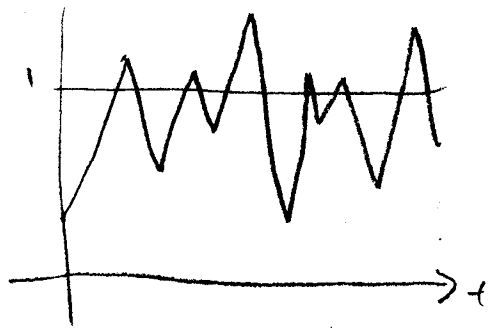
Case 2: $1<r<2 \Rightarrow -1<1-r<0 \Rightarrow p_{t+1} = \underbrace{(1-\lambda)}_{<0} p_t$



$\Rightarrow p_t \to 0$, sign toggles.

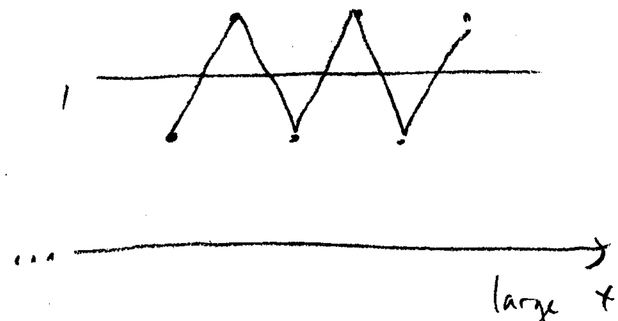
This is "underdamped"

<u>Case 3</u>: $r > 2 \Rightarrow 1-r < -1$ $\quad p_{t+1} \approx \overbrace{(1-r)}^{\|\cdot\|>1} p_t$ ; $\quad p_t \not\to 0$

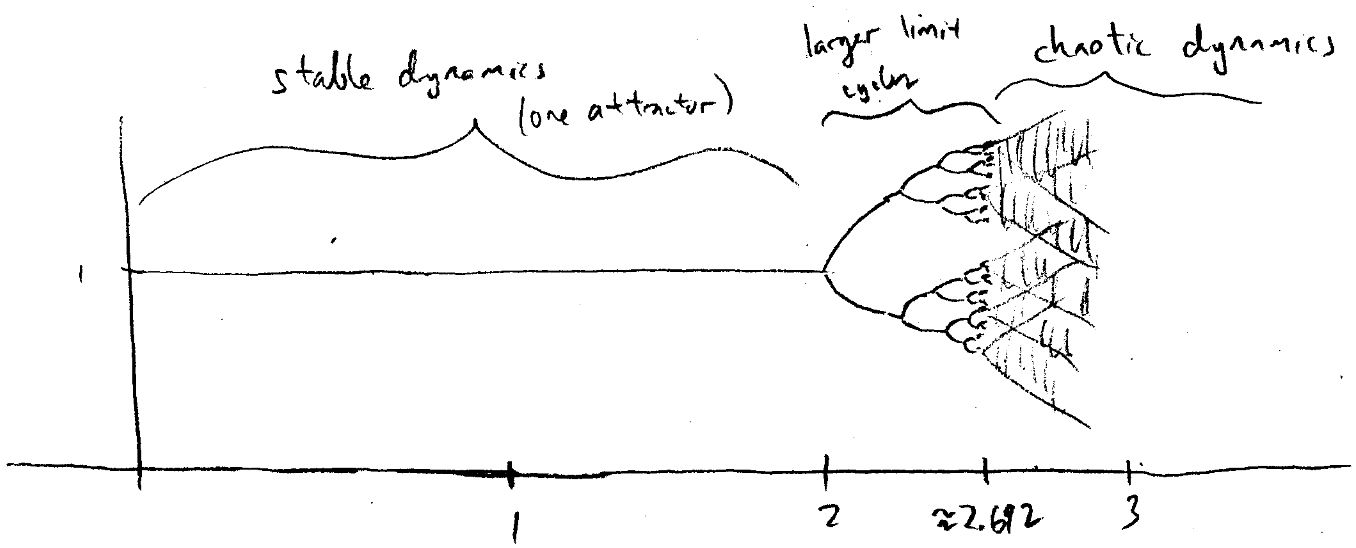



$r > 2.7$ <u>chaotic</u> dynamics

vs.




$r \approx 2$. approaches <u>periodic dynamics</u> in the limit.




Computationally, it's observed:

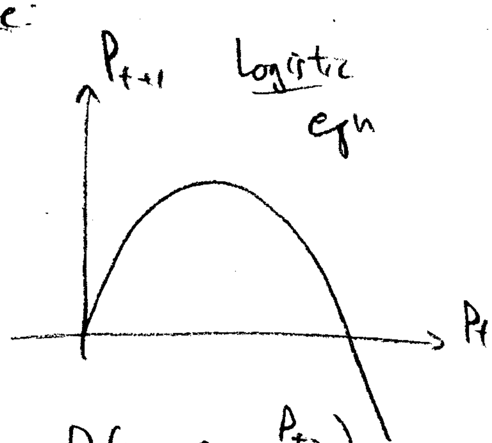
- $\boxed{r < 2}$ <u>Stable</u> dynamics: long-term behavior $\longrightarrow p^* = 1$.
- $\boxed{2 < r < 2.692}$ <u>Unstable</u>, <u>non-chaotic</u>: limit cycles of size $2^n$
- $\boxed{r > 2.692}$ <u>Chaotic</u> dynamics: Tiny perturbations will diverge. No periodic behavior. (ex: butterfly in NZ vs. hurricane in Hawaii)

May, "Science"

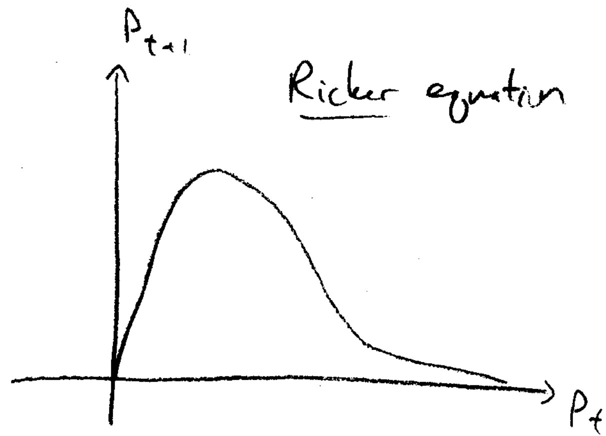
**Fact:** Chaos was first discovered in simple population models in 1978!
Also observed in flour beetles (Cushing et al. 2001).

## Other population models

Compare:



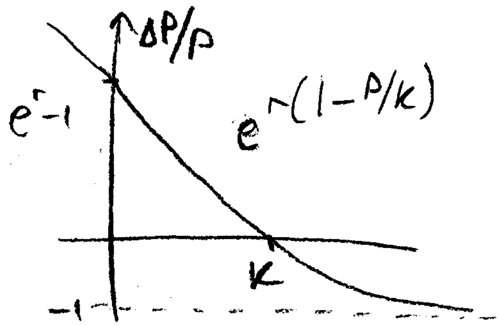

vs.




$$P_{t+1} = P\left(1+r\left(1-\frac{P_t}{k}\right)\right) \qquad P_{t+1} = P_t e^{r(1-P_t/k)}$$

$\frac{\Delta P}{P}$ can't be $< -1$ (per capita death rate can't be $> 1$)

This motivates: $\frac{\Delta P}{P} = a e^{-bP} - 1$




Substitute: $b = \frac{r}{k}$, $a = e^r$

$$\Rightarrow \frac{\Delta P}{P} = e^r e^{-rP/k} - 1 = e^{r(1-P/k)} - 1$$

Algebra $\rightsquigarrow$ $P_{t+1} = P_t e^{r(1-P_t/k)}$

Key: Mathematically capture qualitative graphical features that are realistic.