Linear models of structured populations

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Motivation: Population dynamics

Consider a population divided into several groups, such as

- children and adults
- egg, larva, pupa, adult

For example, consider a population of insects

$$\text{Egg} \longrightarrow \text{Larva} \longrightarrow \text{Adult} \longrightarrow \text{Dead}$$

$E_t = \#$ eggs at time t

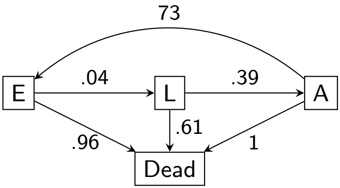
$L_t = \#$ larve at time t

$A_t = \#$ adults at time t

An example

Suppose we have the following data:

- 4% of eggs survive to become larvae
- 39% of larvae make it to adulthood
- The average adult produces 73 eggs each
- Each adult dies after 1 day



We can write this as a system of difference equations:

$$\begin{cases} E_{t+1} = 73A_t \\ L_{t+1} = .04E_t \\ A_{t+1} = .39L_t \end{cases} \qquad \begin{bmatrix} 0 & 0 & 73 \\ .04 & 0 & 0 \\ 0 & .39 & 0 \end{bmatrix} \begin{bmatrix} E_t \\ L_t \\ A_t \end{bmatrix} = \begin{bmatrix} E_{t+1} \\ L_{t+1} \\ A_{t+1} \end{bmatrix}.$$

By back-substitution, or inspection, we can deduce the following:

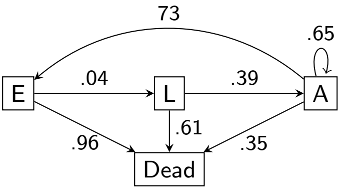
$$A_{t+3} = (.39)(.04)(73)A_t = 1.1388A_t$$

Thus, this is just exponential growth. But what if instead of dying, 65% of adults survive another day?

A slighly more complicated example

Suppose we have the following data:

- 4% of eggs survive to become larvae
- 39% of larvae make it to adulthood
- The average adult produces 73 eggs each
- Each day, 35% of adults die.



This yields a more complicated system of difference equations:

$$\begin{cases} E_{t+1} = 73A_t \\ L_{t+1} = .04E_t \\ A_{t+1} = .39L_t + .65A_t \end{cases} \qquad \begin{bmatrix} 0 & 0 & 73 \\ .04 & 0 & 0 \\ .65 & .39 & 0 \end{bmatrix} \begin{bmatrix} E_t \\ L_t \\ A_t \end{bmatrix} = \begin{bmatrix} E_{t+1} \\ L_{t+1} \\ A_{t+1} \end{bmatrix}.$$

Questions

- Best way to solve this?
- What is the growth rate?
- What is the long-term behavior?
- How much effect does changing the initial conditions have?

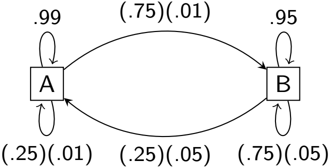
Another example

Consider a forest that has 2 species of trees, A and B. Let A_t and B_t denote the population of each, in year t.

When a tree dies, a new tree grows in its place (either species).

Each year:

- 1% of the A-trees die
- 5% of the B-trees die
- 25% of the vacant spots go to species A
- 75% of the vacant spots go to species B



This can be written as a 2×2 system:

$$\begin{cases} A_{t+1} = .99A_t + (.25)(.01)A_t + (.25)(.05)B_t \\ B_{t+1} = .95B_t + (.75)(.01)A_t + (.75)(.05)B_t \end{cases}$$

$$\begin{bmatrix} .9925 & .0125 \\ .0075 & .9875 \end{bmatrix} \begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} A_{t+1} \\ B_{t+1} \end{bmatrix}$$

Solving systems of difference equations

One way to solve $\mathbf{x}_{t+1} = P\mathbf{x}_t$:

$$\begin{aligned}\mathbf{x}_1 &= P\mathbf{x}_0 \\ \mathbf{x}_2 &= P\mathbf{x}_1 = P(P\mathbf{x}_0) = P^2\mathbf{x}_0 \\ \mathbf{x}_3 &= P\mathbf{x}_2 = P^3\mathbf{x}_0 \\ &\vdots\end{aligned}$$

A better method

Find the *eigenvalues* and *eigenvectors* of P.

Then write the initial vector $\mathbf{x}_0$ using a *basis of eigenvectors*.

Suppose $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then

$$\begin{aligned}\mathbf{x}_1 &= P\mathbf{x}_0 = P(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 \\ \mathbf{x}_2 &= P\mathbf{x}_1 = P^2\mathbf{x}_0 = P(c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2) = c_1\lambda_1^2\mathbf{v}_1 + c_2\lambda_2^2\mathbf{v}_2\,. \\ &\vdots \\ \mathbf{x}_t &= P^t\mathbf{x}_0 = c_1\lambda_1^t\mathbf{v}_1 + c_2\lambda_2^t\mathbf{v}_2\,.\end{aligned}$$

An example, revisted

Let us revisit our "tree example", where $P = \begin{bmatrix} .9925 & .0125 \\ .0075 & .9875 \end{bmatrix}$.

The eigenvalues and eigenvectors of P are

$$\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \qquad \lambda_2 = .98, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Consider the initial condition $\mathbf{x}_0 = \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 10 \\ 990 \end{bmatrix}$.

First step

Write $\mathbf{x}_0 = c_1 \begin{bmatrix} 5 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, i.e., solve $P\mathbf{c} = \mathbf{x}_0$:

$$\begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 990 \end{bmatrix}.$$

$$\mathbf{c} = P^{-1}\mathbf{x}_0 = -\frac{1}{8} \begin{bmatrix} -1 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ 990 \end{bmatrix} = \begin{bmatrix} 125 \\ -615 \end{bmatrix}$$

Thus, our initial vector is $\mathbf{x}_0 = \begin{bmatrix} 10 \\ 990 \end{bmatrix} = 125 \begin{bmatrix} 5 \\ 3 \end{bmatrix} - 615 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

An example (cont.)

Solving for $\mathbf{x}_t$

Once we have written $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, the solution $\mathbf{x}_t$ is simply

$$\mathbf{x}_t = P^t\mathbf{x}_0 = c_1\lambda_1^t\mathbf{v}_1 + c_2\lambda_2^t\mathbf{v}_2\,.$$

In our example, $\mathbf{x}_0 = 125\begin{bmatrix}5\\3\end{bmatrix} - 615\begin{bmatrix}1\\-1\end{bmatrix}$, and so

$$\mathbf{x}_t = 125(1)^t\begin{bmatrix}5\\3\end{bmatrix} - 615(.98)^t\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}625-(615)(.98)^t\\375+(615)(.98)^t\end{bmatrix}.$$

The long-term behavior of this system is

$$\lim_{t\to\infty}\mathbf{x}_t = 125\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}625\\375\end{bmatrix}\,.$$

Notice that this does *not* depend on $\mathbf{x}_0$!