1. Introduction to Modeling:

Let's think back to basic models we've seen, & how to extend them.

- Falling object (physics): $x'' = -9.8$

$$x(t) = -4.9t^2 + Ct + D.$$

- Exponential growth (Biology; Malthus, 1798): $m' = rm$

$$m(t) = Ce^{rt}$$

- Exponential growth (finance): $P' = rP$ $\quad r$ = interest rate, e.g., 0.05

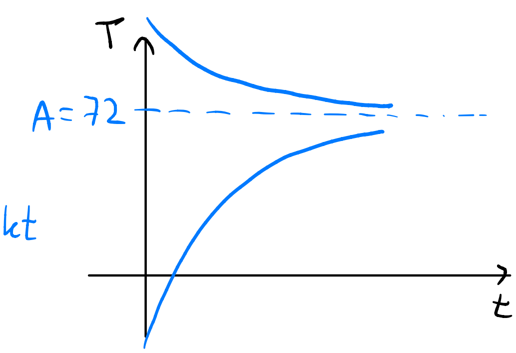
$$P(t) = Ce^{rt}$$

- Exponential decay (chemistry): $m' = -km$

$$m(t) = Ce^{-kt}$$

- Decay to value (Newton's law of cooling): $T' = k(A - T)$

$$T(t) = A + Ce^{-kt}$$



Now, let's modify some of these:

- Falling object w/ air resistance: $F = ma = mv' = -mg + R(v)$

Air resistance force approx. proport. to velocity, in opposite direction

$R(v) = -rv$

So $mv' = -mg - rv \Rightarrow v' = -g - \frac{r}{m}v$

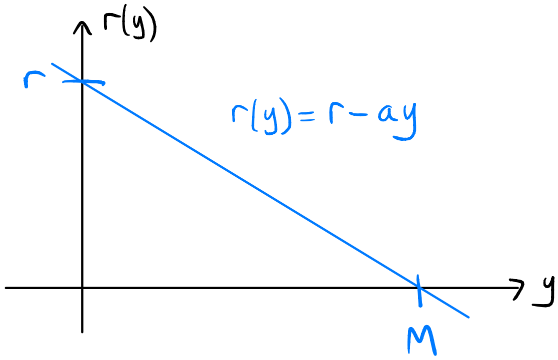
$$\Rightarrow \quad = \frac{r}{m}\left(-\frac{mg}{r} - v\right)$$

$$v(t) = -\frac{mg}{r} + Ce^{\frac{r}{m}t}$$ "decay to a value"

- Logistic model for population growth (Verhulst, 1838)

Expon. growth: $y' = ry$ growth rate r constant

Realistically, growth rate decreases with y



$r = \lim_{y \to 0} r(y)$ "intrinsic growth rate"

M = carrying capacity $= \lim_{t \to \infty} y(t)$

$a = r/M$

Now, $y' = r(y)\, y = \left(r - \frac{r}{M} y\right) y \implies y' = ry\left(1 - \frac{y}{M}\right)$

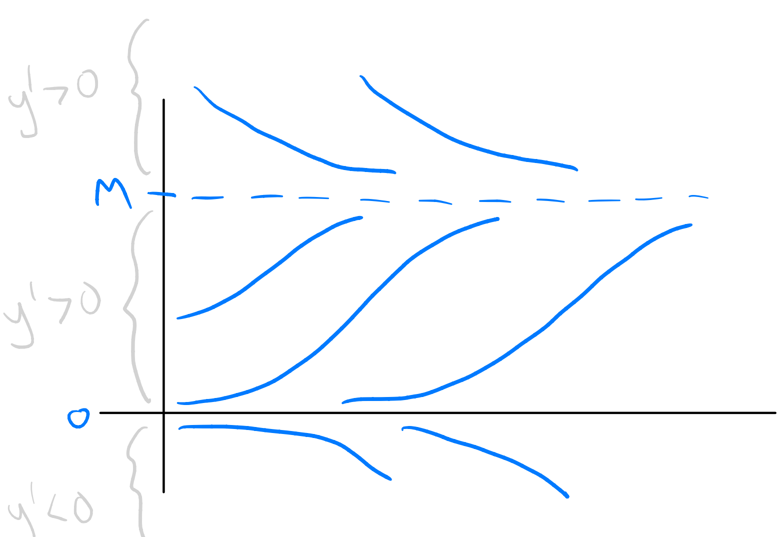
To solve: separate variables: $\frac{dy}{dt} = \frac{r}{M}\, y\,(1 - y)$

$$\implies \int \frac{dy}{y(1-y)} = \int \frac{r}{M} \implies \dots \boxed{y(t) = \frac{M}{1 + Ce^{-rt}}}$$

<u>Steady-states</u>: $y' = 0 \implies y = 0,\ M$

Init pop: $y(0) = \frac{M}{1+C}$

limiting pop: $\lim_{t \to \infty} y(t) = M$

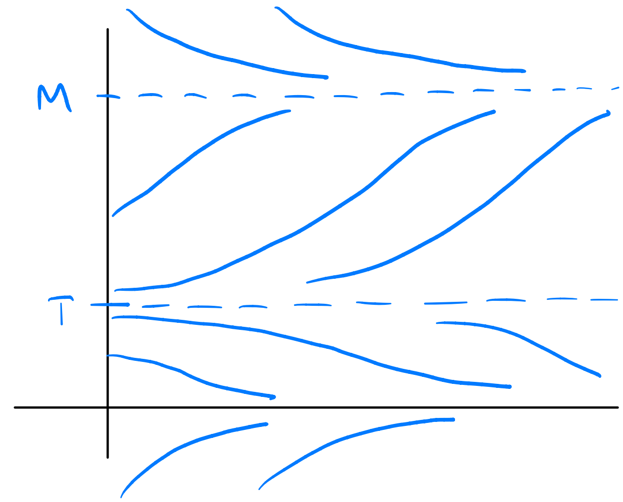


$$y' = r\, y\, \left(1 - \frac{y}{M}\right)$$

	y'		$r\,y$	$\left(1 - \frac{y}{M}\right)$
$y > M$	$-$	$=$	$+$	$-$
$0 < y < M$	$+$	$=$	$+$	$+$
$y < 0$	$-$	$=$	$-$	$+$

<u>Threshold equation:</u> Let's now add an "extinction threshold" T

Want steady-states $y(t) = 0, M, T$



$$y' = -ry\left(1 - \frac{y}{M}\right)\left(1 - \frac{y}{T}\right)$$

$y > M$	$-$	$=$	$-$	$+$	$-$	$-$
$T < y < M$	$+$	$=$	$-$	$+$	$+$	$-$
$0 < y < T$	$-$	$=$	$-$	$+$	$-$	$-$
$y < 0$	$+$	$=$	$-$	$-$	$-$	$-$

This actually modeled the now extinct passenger pigeon quite well.