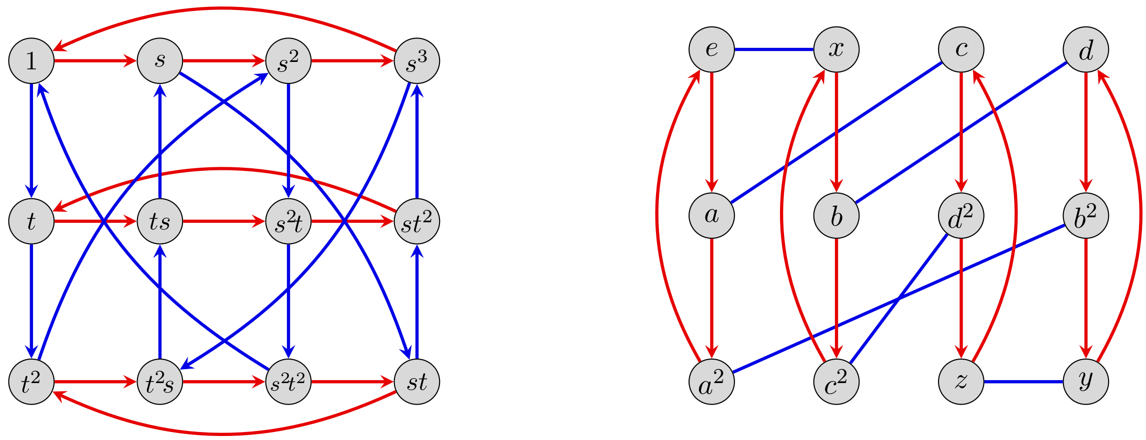
Math 4120, Final Exam. Friday December 10, 2021

1. (20 points) On our very first homework assignment, we saw Cayley diagrams of two unknown groups of order 12:



We now know that there are five groups of this size: C_{12}, $C_6 \times C_2$, D_6, A_4, and Dic_6.

   (a) Determine which groups these are isomorphic to, and justify your answer. You may assume common knowledge about the aforemtioned groups provided you state it.

   (b) Write a presentation for each group, using the generators given.

   (c) On HW 1, you were asked to create a Cayley table for each group (you do *not* need to do that here), and order the elements in the first group by

$$1,\ t,\ t^2,\ s,\ ts,\ t^2s,\ s^2,\ s^2t,\ s^2t^2,\ s^3,\ st^2,\ st$$

   and those in the second by

$$e,\ x,\ y,\ z,\ a,\ b,\ c,\ d,\ a^2,\ b^2,\ c^2,\ d^2.$$

   The last part of the problem said: "*Squint your eyes. Do you see any patterns in these tables?*". Without actually constructing these tables, explain what such patterns would represent, and why. Would you expect to find some, and if so, what would they look like, and what would that tell us about these groups? Justify your answer.

2. (10 points) Let G be a group with a subgroup $H = \langle b, c \rangle$. Be as specific as possible with your answers.

   (a) If $a \in H$, then what is $\langle a, b, c \rangle$?

   (b) If $a \notin H$ and $[G : H] = 2$, then what is $\langle a, b, c \rangle$?

   (c) If $a \notin H$, and $|G| = 48$ and $|H| = 6$, then what do you know about the order of $\langle a, b, c \rangle$?

3. (10 points) Let G be a group of order 30. Show that if G is not cyclic, then it is nonabelian. State any results about the structure of groups that you use.

4. (15 points) Define the homomorphism $\phi\colon D_4 \to V_4$ by $\phi(r) = h$ and $\phi(f) = v$.

   (a) Find the images of the remaining six elements of D_4.

   (b) Is ϕ an embedding or a quotient map?

   (c) Find $\mathrm{Ker}(\phi)$ and $\mathrm{Im}(\phi)$.

   (d) What does the *fundamental homomorphism theorem* tell us about ϕ? (Your answer should be more specific than just stating the FHT.)

5. (30 points) Let G be a finite group.

   (a) Show that if $H \leq G$ and $[G : H] = 2$, then $H \trianglelefteq G$. Does this also hold for infinite groups?

   (b) Formally define the following terms: (i) an *action* of a group G on a set S, (ii) the *orbit* of $s \in S$, (iii) the *stabilizer* of $s \in S$.

   (c) Carefully state the orbit-stabilizer theorem.

   (d) Let G act on itself by conjugation, that is, $S = G$ and we have a homomorphism $\phi\colon G \to \mathrm{Perm}(S)$, where

   $$\phi(g) = \text{the permutation sending each } x \text{ to } g^{-1}xg.$$

   Describe $\mathrm{orb}(s)$, $\mathrm{stab}(s)$, $\mathrm{fix}(g)$, $\mathrm{Ker}(\phi)$, and $\mathrm{Fix}(\phi)$, using familiar group-theoretic terms.

   (e) Show that the size of any conjugacy class divides $|G|$.

   (f) Show that if a group G contains an element $x \in G$ that has exactly two conjugates (itself included), then G is not simple. [*Hint*: The previous parts of the problem are helpful!]

6. (15 points) Let G be a group of order $4120 = 2^3 \cdot 5 \cdot 103$. Answer the following:

   (a) How big is a Sylow 2-subgroup of G? How big is a Sylow 103-subgroup?

   (b) Use the Sylow theorems to show that G cannot be simple.

7. (10 points) Draw the subgroup lattice of the dihedral group D_7. Label the edges by the corresponding index, and denote the conjugacy classes by dashed circles.

8. (10 points) How big is a Sylow 2-subgroup of D_n when n is odd? Describe all of them explicitly by their generator(s). How many are there? [*Hint*: If you're stuck, draw a picture of an n-gon!]

9. (20 points) Let $H, N \leq G$ and suppose that $N \trianglelefteq G$. Show that

   $$H/(H \cap N) \cong HN/N.$$

   You may assume that $HN \leq G$, and that both $N \trianglelefteq HN$ and $H \cap N \trianglelefteq H$. [*Hint*: Start with a map φ from H. Make sure you write down how it's defined.]

10. (20 points) Give an example of each of the following (no justification needed).

    (a) An element $g \in G$ of order 2 in an infinite group.
    (b) A nonabelian group such that $Z(G) = \{e\}$.
    (c) An nonabelian group such that every subgroup is normal.
    (d) A group G of order 16 such that $g^2 = e$ for all $g \in G$.
    (e) A subgroup of $\mathbb{Z}_{24}$ of order 4.
    (f) Two nonisomorphic abelian groups of order 24, neither of which are cyclic.
    (g) Two non-isomorphic subgroups of D_4 of the same order. [Give generating sets.]
    (h) A chain of subgroups $K \trianglelefteq H \trianglelefteq G$ for which $K \ntrianglelefteq G$. [Give generating sets.]
    (i) A subgroup H of a group G such that $N_G(H) = H$.
    (j) A subgroup H of a group G such that $H \lneq N_G(H) \lneq G$.
    (k) A generating set for S_5 that is *minimal*, but not minimum in size.
    (l) Two elements in S_5 of the same order that are not conjugate.
    (m) A group presentation for the dihedral group D_4.
    (n) An integral domain that is not a field.
    (o) A nonzero element of a ring R that is neither a unit nor a zero divisor.
    (p) A ring with infinitely many units.
    (q) A ring with infinitely many zero divisors.
    (r) A finite field.
    (s) A maximal ideal of the polynomial ring $\mathbb{Z}[x]$.
    (t) A prime ideal $I \neq R$ that is not maximal. (Make sure you specify what R is.)

11. (20 points) Let R be a ring with 1.

    (a) Define what it means for a set $I \subseteq R$ to be a *left ideal* of R.
    (b) Show that if I contains a unit (an element u with a multiplicative inverse), then $I = R$.
    (c) Let $\phi\colon R \to S$ be a ring homomorphism. Show that $\operatorname{Ker}\phi := \{r \in R \mid \phi(r) = 0\}$ is a left ideal of R. [It is actually a two-sided ideal, but you don't need to show that.]

12. (10 points) Let I be an ideal of a commutative ring R with 1. Prove <u>one</u> of the two statements.

    (i) I is maximal if and only if R/I is a field.
    (ii) I is prime if and only if R/I is an integral domain.

13. (10 points) What was your favorite topic in this class? Specifically, what did you find the most interesting, and why?