Math 4120, Final Exam. December 13, 2022

1. (8 points) Complete the following *formal* mathematical definitions. For full credit, propertly use terminology like ∀ ("for all") or ∃ ("there exists"), where appropriate.

   (a) Define what a *homomorphism* ϕ is between two groups, G and H.

   (b) Define the *kernel* of a homomorphism:

   $$\mathrm{Ker}(\phi) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad \Big\}.$$

   (c) Define an *action* ϕ of a group G on a set S:

   (d) Define the *kernel* of a group action, and write $s.\phi(g)$ for "*the image of s under the permutation $\phi(g)$*":

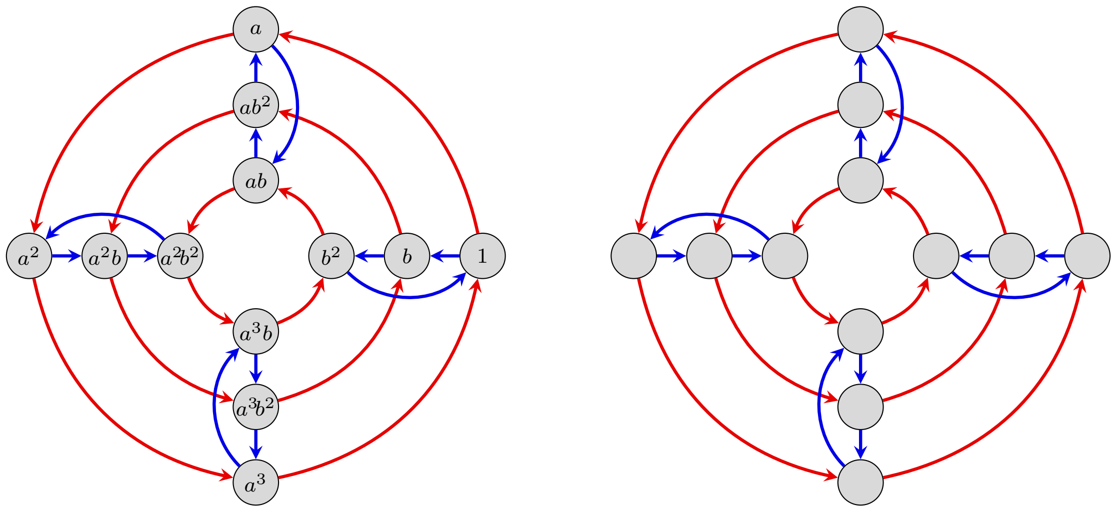
   $$\mathrm{Ker}(\phi) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad \Big\}.$$

2. (8 points) Let $\phi\colon G \to H$ be a homomorphism. Show that

   $$\mathrm{Im}(\phi) := \big\{\phi(g) \mid g \in G\big\}$$

   is a subgroup of H.

3. (36 points) The Cayley graph of a group $G = \langle a, b \rangle$ of order 12 is shown below.



(a) Write the *order* of each element in the corresponding node in the blank graph.

(b) Write down a presentation for this group.

(c) Find all left cosets of $A = \langle a \rangle$, then find all right cosets. Write them as subsets.

(d) Find all left cosets of $B = \langle b \rangle$, then find all right cosets. Write them as subsets.

(e) Find the normalizers $N_G(A)$ and $N_G(B)$.

(f) Are either A or B normal? Why or why not?

(g) Find the conjugacy classes, $\mathrm{cl}_G(A)$ and $\mathrm{cl}_G(B)$, of these subgroups.

(h) Using *only* the information that $|G| = 12 = 2^2 \cdot 3$, determine the order of a Sylow 2-subgroup, and the order of a Sylow 3-subgroup.

(i) How many Sylow 2-subgroups does G have? How many Sylow 3-subgroups? Justify your answer.

(j) Is G a simple group? Why or why not?

(k) Find the center, $Z(G)$.

(l) Construct the subgroup lattice. Write the subgroups by their generator(s).

Order $= 12$ $G = \langle a, b \rangle$

6

4

3

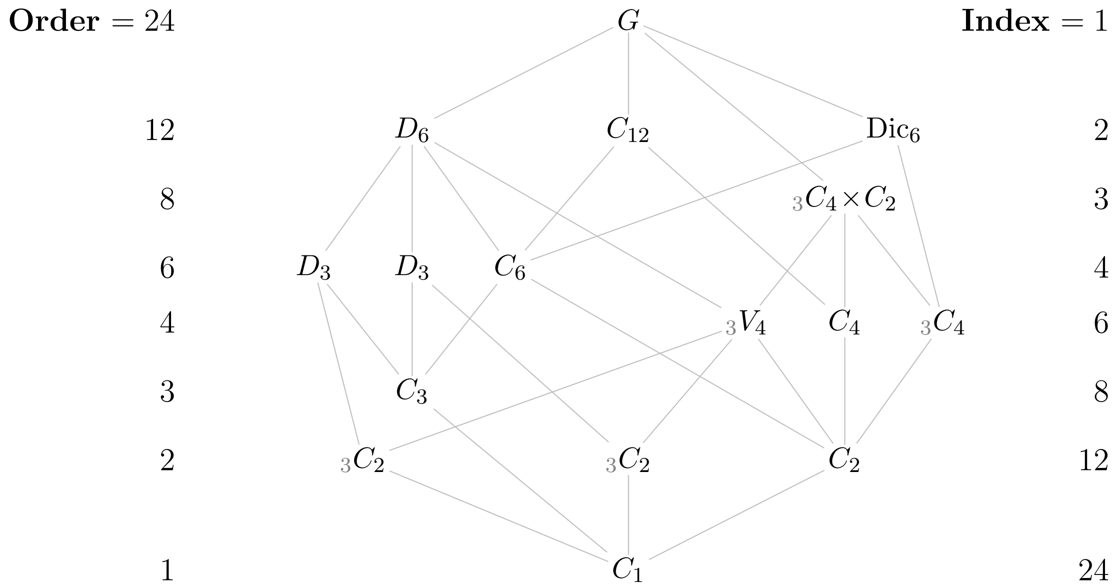
2

1 $\langle 1 \rangle$

(m) There are five groups of order 12: C_{12}, $C_6 \times C_2$, D_6, Dic_6, and A_4. Which group is this? Briefly justify your answer for full credit.

(n) Is G isomorphic to a direct or semidirect product of nontrivial subgroups? Why or why not?

4. (30 points) Answer questions about the following group, whose subgroup lattice is shown below.



(a) G has ______________ subgroup, which fall into ______________ conjugacy classes.

(b) G has exactly ____ normal subgroups.

(c) G has ___________ subgroup(s) of order 2 and ___________ element(s) of order 2.

(d) G has ___________ subgroup(s) of order 3 and ___________ element(s) of order 3.

(e) G has ______________ subgroup(s) of order 4, of which ______________ are cyclic.

(f) Find three distinct pairs of subgroups, $H, K \leq G$ that have quotient $H/K \cong V_4$.

(g) Each non-normal order-2 subgroup has a normalizer isomorphic to ______________.

(h) Each D_3 subgroup has a normalizer isomorphic to _____________________________.

(i) This group has a quotient G/C_4 isomorphic to _____. [*Hint*: Determine the order, then count the index-2 subgroups.]

(j) This group has a quotient G/C_2 isomorphic to _____. [*Hint*: Same as above.]

(k) The quotient G/C_3 is isomorphic to _____. [*Hint*: Determine the order. Which lattice do you see?]

(l) The commutator subgroup is $G' \cong$ _____, and the abelianization is $G/G' \cong$ ________.

(m) There are $n_2 =$ _____ Sylow 2-subgroups, which are isomorphic to ________.

(n) There are $n_3 =$ _____ Sylow 3-subgroups, which are isomorphic to ________.

(o) The largest order of an element in G is _____, and there are _____ element(s) of that order.

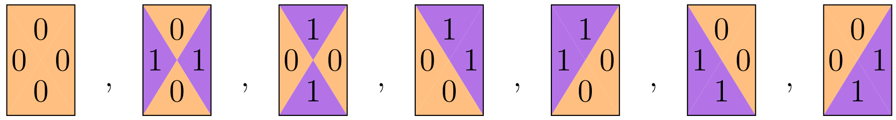
(p) Write G as a (nontrivial) direct product of two subgroups, in as many distinct ways as possible.

(q) Write G as a (nontrivial) direct semidirect product of two subgroups, in as many distinct ways as possible.

5. (8 points) Suppose $H \leq G$ is the only subgroup of order m. Prove that H is normal.

6. (10 points) Use the Sylow theorems to show that there are no simple groups of order pq, where $p < q$ are distinct primes. Clearly state what result(s) you are using.

7. (15 points) Consider the following set of "binary rectangles":

$S = \{$  $\}$

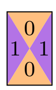
The Klein-4 subgroup $H = \{1, r^2, rf, r^3f\}$ of D_4 acts on S via $\phi\colon H \to \text{Perm}(S)$, where

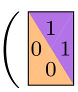
$\phi(r^2)$ = rotates each tile by 180°

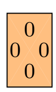
$\phi(rf)$ = swaps the *digits* on each tile across the "positively sloped" diagonal axis.

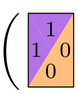
(a) Pick a minimal generating set and then draw the *action graph*. (Feel free to label the rectangles above A,B,C,D,E,F, to save time.)

(b) Find the following:

- stab() =

- stab() =

- stab() =

- stab() =

- $\text{fix}(1) =$
- $\text{fix}(rf) =$
- $\text{fix}(r^2) =$
- $\text{fix}(r^3f) =$
- This action has ________ orbits, which by the orbit-counting theorem, is also equal to the average ____.
- $\text{Fix}(\phi) =$
- $\text{Ker}(\phi) =$

8. (25 points) Fill in the following blanks.

1. The smallest non-cyclic group is ______________________.
2. The smallest group with $Z(G) \neq G$ is ______________________.
3. The group $A \times B$ has *at least* __________ normal subgroups (assume $|A|, |B| > 1$).
4. There are more groups of order exactly $n =$__________, than of any other $n < 1000$.
5. $xH = yH$ if and only if $y^{-1}x$ is ______________________.
6. The subgroup $\langle (12), (34) \rangle$ of S_5 is isomorphic to ______________________.
7. Up to isomorphism, there are ______________ abelian group(s) of order 30.
8. An example of a minimal generating set of S_5 of *maximal* size is ______________.
9. An example of a minimal generating set of S_5 of *minimum* size is ______________.
10. The bin. op. on G/N is well-defined if $aN = bN$ and $cN = dN$, implies __________.
11. If G acts on its subgroups by conjugation, $H \in \text{Fix}(\phi)$ if and only if ____________.
12. If Q_8 acts on its subgroups by conjugation, then $\text{Ker}(\phi) =$ ______________.
13. A group H is a p-subgroup of G if and only if ______________________.
14. The second Sylow theorem says that all Sylow p-subgroups are ______________.
15. A nontrivial proper ideal I of a ring cannot contain any ______________.
16. If R is commutative, R/I is a field if and only if I is ______________________.
17. An example of a subring that is not an ideal is ______________________.
18. A maximal ideal of $\mathbb{Z}[x]$ is ______________________.
19. A non-maximal prime ideal of $\mathbb{Z}[x]$ is ______________________.
20. The finite field $\mathbb{F}_{16}$ contains ______________________ units.
21. The additive group of the field $\mathbb{F}_{16}$ is isomorphic to ______________________.
22. The multiplictive group of the field $\mathbb{F}_{16}$ is isomorphic to ______________.
23. Zorn's lemma is useful for showing that every $r \in R$ is contained in ____________.
24. An example of an integral domain that is not a field is ______________.
25. An example of commutative ring that is not an integral domain is ____________.

9. (20 points) Let I be an ideal of a commutative ring R with 1.

(a) The quotient ring consists of the set $R/I := \{ \qquad\qquad\qquad\qquad \}$.

(b) The additive identity is __________, and the muliplicative identity is __________.

(c) Write down how addition and multiplication (of cosets) are defined in the quotient ring.

(d) Carefully define what it means for an element (coset) of R/I to be a *zero divisor*.

(e) Define what it means for I to be a *prime ideal* of R.

(f) Prove that I is prime if and only if R/I is an integral domain.

10. (20 points) Suppose $A, B \leq G$ and A normalizes B.

(a) Show that $B \trianglelefteq AB$.

(b) You may assume that $(A \cap B) \trianglelefteq A$. Prove the *diamond theorem*:

$$A/(A \cap B) \cong AB/B.$$

[*Hint*: Start by defining a explicit map $\phi\colon A \to AB/B$.]

(c) Prove the diamond theorem for rings: if S is a subgroup and I an ideal, then

$$S/(S \cap I) \cong (S + I)/I.$$

You may assume that $S \cap I$ is an ideal of S, and I is an ideal of $(S + I)$. [*Hint*: This should be very short. The key understanding just *what* you have to prove. Start with the map from Part(b), slightly modified because S and I are *additive* groups.]

11. (8 points) Make a list of all abelian groups of order $108 = 2^2 \cdot 3^3$, up to isomorphism. That is, each group should appear exactly once on your list.

12. (8 points) Draw the *subring lattice* of $\mathbb{Z}_2^2 = \{00, 01, 10, 11\}$. Write the subgroups by generator(s). Then determine which of them are (i) ideals (circle these), (ii) subrings but not ideals (underline these), (iii) subgroups but not subrings (put an X through these).

13. (4 points) What was your favorite topic in this class? Specifically, what did you find the most interesting, and why?