Math 8510, Midterm 2. November 30, 2022

1. (6 points) Complete the following sentences of formal mathematical definitions. Make sure you use terminology like $\forall$ ("for all") or $\exists$ ("there exists"), where appropriate.

   (a) A *ring* R is...

   (b) A *zero divisor* of a commutative ring R is...

   (c) A *left ideal* I of a ring R is...

   (d) A *homomorphism* ϕ from a ring R to S is...

   (e) The *kernel* of a ring homomorphism ϕ from R is...

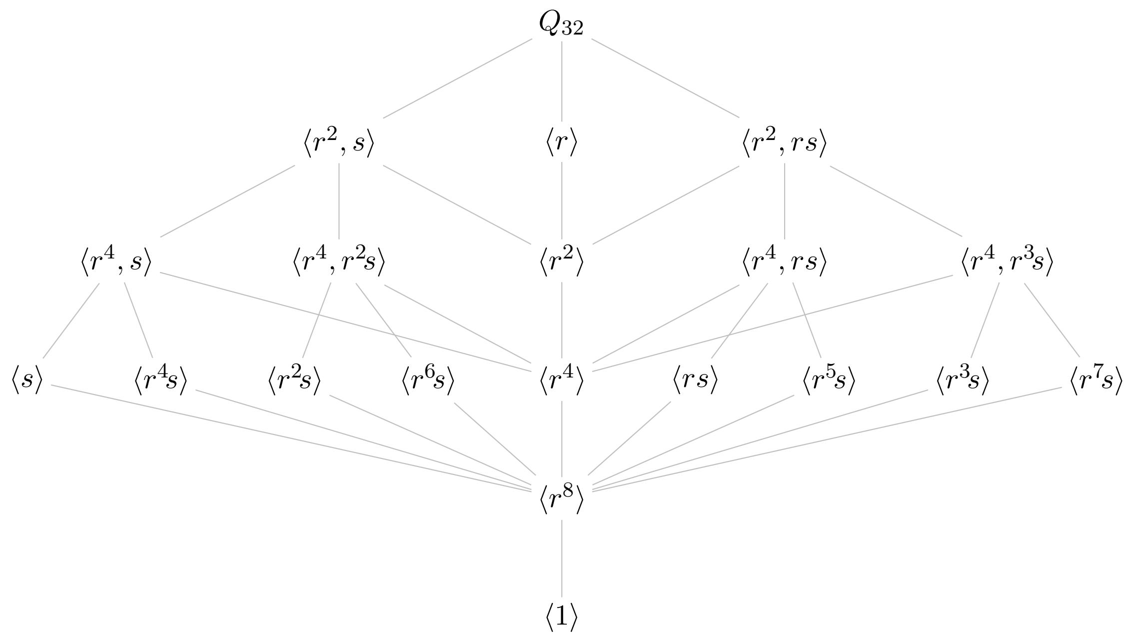
   (f) An ideal P of a commutative ring R is *prime* if ...

2. (8 points) Let $\mathbb{Z}[x]$ and $\mathbb{Z}[[x]]$ denote the rings of polynomials, and formal power series, respectively:

   $$\mathbb{Z}[x] = \big\{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{Z},\ n \in \mathbb{Z}\big\}, \qquad \mathbb{Z}[[x]] = \big\{a_0 + a_1 x + a_2 x^2 + \cdots \mid a_i \in \mathbb{Z}\big\}.$$

   Do you expect either of these rings to be isomorphic to a *product* in the category **Ring**? What about a *co-product*? You do *not* need to prove anything – your answer should be short, informal, but convincing.

3. (16 points) The subgroup lattice of the generalized quaternion group $G = Q_{32}$ is shown below.



(a) Determine the isomorphism types of the following quotient groups:

i. $G/\langle r\rangle \cong$ ii. $G/\langle r^2\rangle \cong$ iii. $G/\langle r^4\rangle \cong$ iv. $G/\langle r^8\rangle \cong$

(b) Annotate the ascending central series $\langle 1\rangle = Z_0 \trianglelefteq Z_1 \trianglelefteq \cdots$ and descending central series $Q_{32} = L_0 \trianglerighteq L_1 \trianglerighteq \cdots$ on the subgroup lattice. Then fill in the blanks for each distinct subgroup (i.e., until the series stabilizes.) Don't just write the definition of Z_i and L_i.

$Z_1 =$ ____________ because ____.

$Z_2 =$ ____________ because ____.

$\vdots$

$L_1 =$ ____________ because ____.

$L_2 =$ ____________ because ____.

$\vdots$

(c) Is Q_{32} nilpotent? Why or why not?

4. (8 points) Show that if a left ideal I of R contains a unit, then $I = R$.

5. (8 points) Show that every field is a simple ring.

6. (12 points) Let $\phi \colon R \to S$ be a ring homomorphism.

   (a) Show that $\operatorname{Ker}(\phi)$ is an ideal of R.

   (b) Prove the fundamental homomorphism theorem: $R/\operatorname{Ker}(\phi) \cong \operatorname{Im}(\phi)$.

   You can and should assume *all* results from group theory, e.g., the FHT for groups. [*Remark*: Both parts should be very short. One aspect of this problem is recognizing and understanding what you have to prove.]

7. (12 points) Let S, S_1, and S_2 be nonempty sets.

   (a) Formally define what it means for a group F to be *free* on S, by a co-universal property. Include a commutative diagram that illustrates this.

   (b) Prove that any two free groups on S are isomorphic.

   (c) For $i = 1, 2$, let F_i be a free group on S_i. Prove or disprove: The group $F = F_1 \times F_2$ is a free group on the set $S = (S_1 \times \{1\}) \cup (\{1\} \times S_2)$.

8. (20 points) Fill in the following blanks.

    1. The smallest non-nilpotent group is ________________________.
    2. In the category of sets, suppose that $|A| = 6$ and $|B| = 10$. Then the size of their *product* is ________________, and the size of their *co-product* is ________________.
    3. Every subgroup of a free group is ________________________.
    4. The order of the co-product of $\mathbb{Z}_4$ with $\mathbb{Z}_6$ is ________________________.
    5. A group presentation for the free product of $\mathbb{Z}_2 = \langle a \rangle$ with $D_3 = \langle r, f \rangle$ is

       ____.
    6. The co-product of $\mathbb{Z}$ with $\mathbb{Z}$ is ____________ in **Grp** and ____________ in **Ab**.
    7. A *functor* is a structure-preserving map between two ________________________.
    8. The group $\mathbb{Z} \times \mathbb{Z}$ is the quotient of the free group F_S on $S = \{a, b\}$ by the smallest normal subgroup containing ________________________.
    9. The ring $\mathbb{Z}$ has ____ unit(s), ____ zero divisor(s), and ____ element(s) that are neither.
    10. In $\mathbb{Z}$, the principal ideal $I = (a, b)$ is generated by $k =$________________________.
    11. The ideal $I = (4)$ in the polynomial ring $R = \mathbb{Q}[x]$ is ________________________.
    12. A commutative ring is an integral domain iff the zero ideal is ________________________.
    13. An ideal I of R is maximal iff for all ideals J satisfying $I \subseteq J \subseteq R$, ________________.
    14. The quotient R/I of a commutative ring is a field if and only if ____________________.
    15. An example of a (non-zero) non-maximal prime ideal is ________________________.
    16. If $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ are groups with $S_1 \supseteq S_2$ and $R_1 \subseteq R_2$, then

        ____.

9. (10 points) Prove the co-universal property for quotients:

   Let G and H be groups, $N \trianglelefteq G$, and $\pi\colon G \twoheadrightarrow G/N$ the canonical quotient map. If $f\colon G \to H$ is a homomorphism whose kernel contains N, then there exists a unique homomorphism $h\colon G/N \to H$ such that $h \circ \pi = f$.

   Include a commutative diagram that illustrates this.

10. (Extra credit, 1 point) With the birth of my son Felix exactly six weeks ago today, my family became a group of order 4. What prominent historical mathematician shares his name?