Math 4120, Final Exam. Friday May 6, 2022

1. (10 points) Complete the following *formal* mathematical definitions.

   (a) A *group* is a set G with an associative binary operation $*$ satisfying ...

   (b) A *ring homomorphism* is...

   (c) A *left ideal* of a ring is...

   (d) An element $p \in R$ in an integral domain is *prime* if ...

   For (a), don't just state the three properties of a group; define what they actually mean, using $\forall$ and $\exists$ where appropriate.

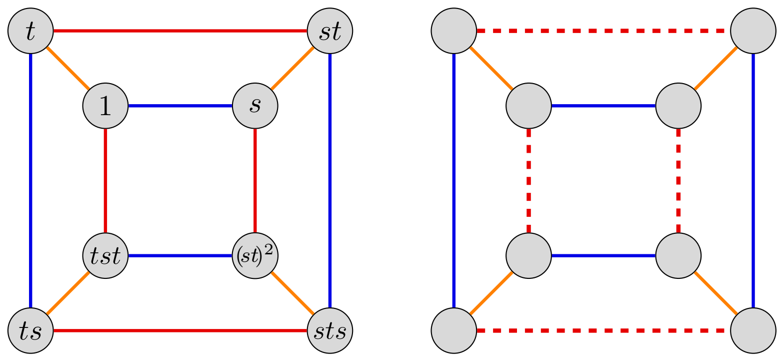
2. (8 points) Consider the curious abstract binary operation $*$ on the set $G = \{1, 2, 3, 4, 5, 6, 7\}$, defined as follows:

$*$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	1	5	4	7	6
3	3	1	2	6	7	4	5
4	4	5	6	7	2	3	1
5	5	4	7	2	6	1	3
6	6	7	4	3	1	5	2
7	7	6	5	1	3	2	4

   Does this binary operation make G into a group? If yes, then determine what familiar group it is isomorphic to. If no, then explain why it fails. In either case, justify your answer. The dashed lines are included just to highlight structure.

3. (30 points) Consider $G = \langle s, t \rangle$, whose Cayley diagram appears below.

G



$\langle 1 \rangle$

(a) On the blank diagram, write the *order* of each element in its corresponding node.

(b) Find the left cosets of $\langle t \rangle$. Write them as subsets of G.

(c) Find the right cosets of $\langle t \rangle$. Write them as subsets of G.

(d) Find the normalizer of $\langle t \rangle$.

(e) How many (distinct) conjugate subgroups does $\langle t \rangle$ have? List them all.

(f) Is $\langle t \rangle$ normal? Why or why not?

(g) Complete the subgroup lattice of G, at top-right. Write subgroups by generator(s).

(h) What familiar group is G isomorphic to? Justify your answer.

(i) For the homomorphism $\phi\colon G \to V_4$ defined by $\phi(s) = h$ and $\phi(t) = v$,

$\phi(1) =$, $\phi(st) =$, $\phi(ts) =$, $\phi(sts) =$, $\phi(tst) =$, $\phi((st)^2) =$

Each answer should be either e, h, v, or hv.

(j) Find $\mathrm{Ker}(\phi)$.

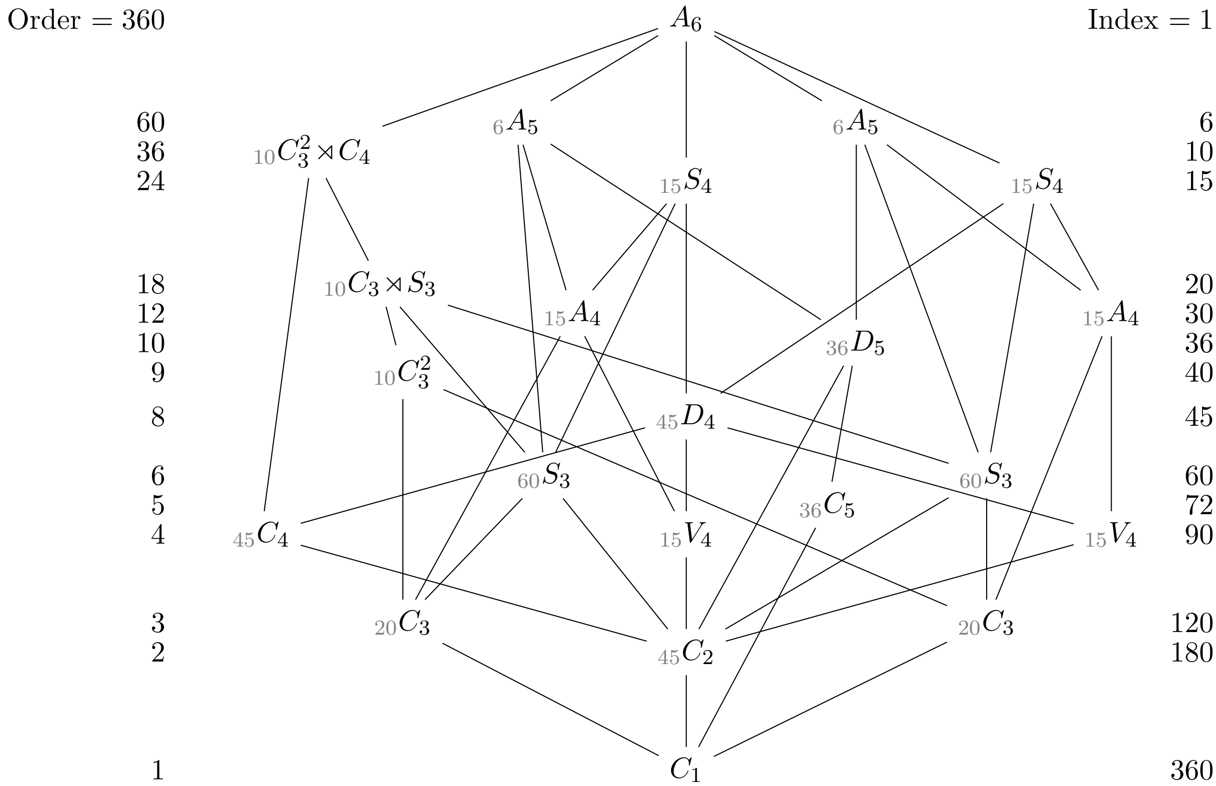
(k) Is ϕ one-to-one? Is it onto?

(l) What does the FHT tell us about these groups? Don't say much, but be specific.

4. (24 points) Fill in the following blanks.

1. The group $\mathbb{Z}_n$ is generated by k if and only if ______________________________.
2. The smallest nonabelian group is ______________________________.
3. The smallest two non-isomorphic groups of the same order are _______ and _______.
4. Over 99% of the groups of order < 2000 have ______________________________.
5. $Z(G) = G$ if and only if ______________________________.
6. If $d \mid n$, then the subgroup $\langle d \rangle$ of $\mathbb{Z}_n$ has order ______________________________.
7. The size of $\mathrm{cl}(f)$ in D_n is ____________ if n is even and ____________ if n is odd.
8. The conjugacy class of $g \in G$ has size 1 if and only if ______________________________.
9. The conjugacy class of $H \leq G$ has size 1 if and only if ______________________________.
10. The group D_6 has ____________ elements of order 3 and ____________ of order 4.
11. The largest order of an element in S_5 is ______________________________.
12. There are exactly ______________________________ odd permutations in S_5.
13. Assuming $n \geq 2$, the quotient S_n/A_n is isomorphic to ______________________________.
14. A homomorphism $\phi \colon G \to H$ is 1-to-1 iff $\phi(g) = 1_H$ implies ____________________.
15. In the quotient group G/H, the product $aH \cdot bH$ is equal to ____________________.
16. The commutator subgroup G' is $\{e\}$ if and only if ______________________________.
17. The group $\mathrm{Aut}(\mathbb{Z})$ has order ______________________________.
18. If G acts on itself by right-multiplication, and $|G| = n$, the action has _____ orbit(s).
19. Let $H \leq G$ and $[G : H] = n$. If G acts on the right cosets of H by right-multiplication, then the action has ______________________________ orbit(s).
20. If R is commutative, I is maximal if and only if R/I is ______________________________.
21. If R is commutative, I is prime if and only if R/I is ______________________________.

5. (28 points) A_6 is the group of even permutations of $\{1,2,3,4,5,6\}$. It is third smallest nonabelian simple group, and its reduced subgroup lattice is shown below.



(a) What is the center of this group?

(b) Of the 501 subgroups of A_6, how many of them are normal?

(c) Of these 501 subgroups, how many are equal to their normalizer? Underline these.

(d) Find a subgroup H for which $H \lneq N(H) \lneq N(N(H)) = N(N(N(H)))$.

(e) Find a subgroup H for which $N^{(k)}(H) := N(\cdots(N(H))) \neq A_6$, for all k.

(f) For each prime p dividing $|A_6| = 360 = 2^3 \cdot 3^2 \cdot 5$, determine how many Sylow p-subgroups there are, and what they are isomorphic to.

(g) Find the commutator subgroup A_6', and the abelianization, A_6/A_6'.

(h) What familiar group is $\langle(123)\rangle$ isomorphic to?

(i) What is the normalizer of $\langle(123)\rangle$ isomorphic to?

(j) What is $\langle(123),(12)(34)\rangle$ isomorphic to?

(k) What is $\langle(123)(456)\rangle$ isomorphic to?

(l) What is $\langle(123),(456)\rangle$ isomorphic to?

(m) Write down a subgroup of A_6 isomorphic to V_4 in terms of *generators* (i.e., permutations, in cycle notation).

6. (15 points) Consider the following set of "binary rectangles":

$$S = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

The Klein 4-group $V_4 = \{e, h, v, hv\}$ acts on S via $\phi\colon V_4 \to \mathrm{Perm}(S)$, where

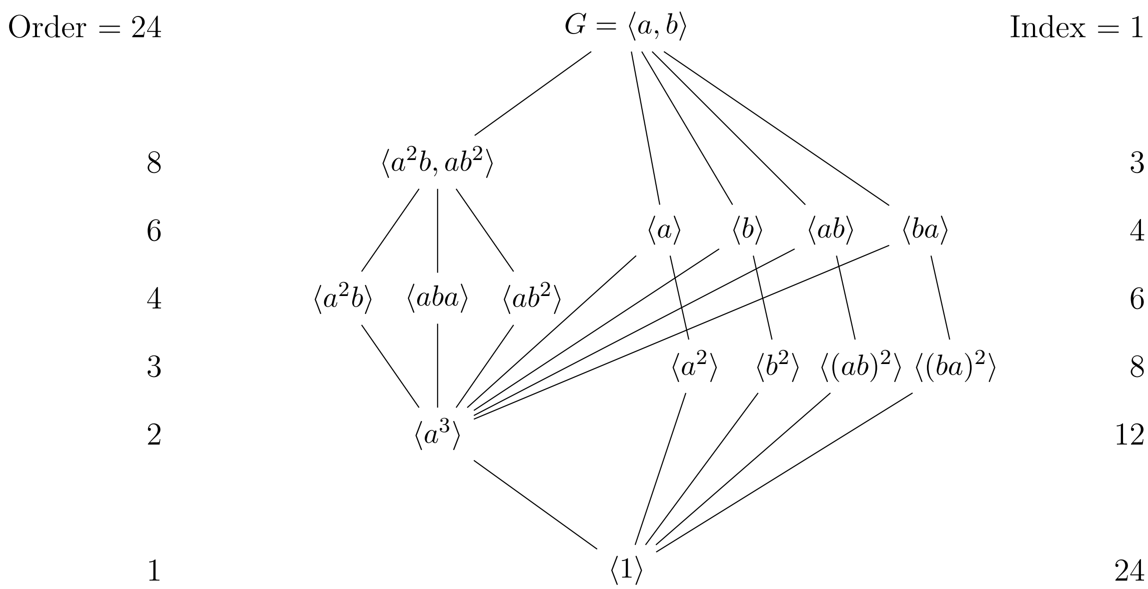
$\phi(h)$ = reflects each tile horizontally (i.e., across its central vertical axis)
$\phi(v)$ = reflects each tile vertically (i.e., across its central horizontal axis).

(a) Draw the *action diagram*.

(b) Find the following:

- $\mathrm{stab}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) =$
- $\mathrm{stab}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) =$
- $\mathrm{stab}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) =$
- $\mathrm{stab}\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) =$
- $\mathrm{fix}(h) =$
- $\mathrm{fix}(v) =$
- $\mathrm{fix}(hv) =$
- $\mathrm{fix}(1) =$
- This action has _________ orbits, which by the orbit-countring theorem, is also equal to the average ____.
- $\mathrm{Fix}(\phi) =$
- $\mathrm{Ker}(\phi) =$

7. (18 points) Let G be the group of order $24 = 2^3 \cdot 3$. whose subgroup lattice is below.



(a) Find the commutator subgroup G' and the abelianization, G/G'.

(b) Find the 2nd and 3rd commutator subgroups, $G'' := (G')'$ and $G''' := (G'')'$.

(c) What familiar group is the quotient $G/\langle a^3 \rangle$ isomorphic to? Justify your answer.

(d) Partition the 15 subgroups into equivalence classes by conjugacy (circle them). Fully justify your answer. Besides each one, write the isomorphism type of its members.

(e) What familiar group is the quotient $\langle a^2b, ab^2 \rangle / \langle a^3 \rangle$ isomorphic to?

(f) Write G as a direct and/or semidirect product of its proper subgroups in as many ways possible, by isomorphism type (not by generators).

(g) How many Sylow 2-subgroups does G have? How many Sylow 3-subgroups?

(h) Is G simple? Why or why not?

8. (15 points) Recall that the *normalizer* of a subgroup $H \leq G$ is

$$N_G(H) = \big\{g \in G \mid gH = Hg\big\} = \big\{g \in G \mid gHg^{-1} = H\big\} = \big\{g \in G \mid ghg^{-1} \in H,\ \forall h \in H\big\}.$$

(a) Prove that $N_G(H)$ is a subgroup of G.

(b) One of the following is true, and the other is false. Prove the true statement, and give a counterexample to the false statement.

i. $H \trianglelefteq N_G(H)$

ii. $N_G(H) \trianglelefteq G$.

(c) Give an example of a group G and a subgroup H that has a coset for which $xH = Hx$ holds, despite $xh \neq hx$ not holding elementwise for every $h \in H$.

9. (10 points) Prove the *fundamental homomorphism theorem for rings*: if $\phi\colon R \to S$ is a ring homomorphism, then the quotient ring $R/\operatorname{Ker}(\phi)$ is isomorphic to $\operatorname{Im}(\phi)$. You may assume the FHT for groups, i.e., that the map

$$\iota\colon R/\operatorname{Ker}(\phi) \longrightarrow \operatorname{Im}(\phi), \qquad \iota(r+I) = \phi(r)$$

is a *group isomorphism*, where $I = \operatorname{Ker}(\phi)$. That is, you *only* need to prove that (i) $\operatorname{Ker}(\phi)$ is an ideal, and that (ii) the group homomorphism ι is also a ring homomorphism.

10. (8 points) Show that a commutative ring R is an integral domain if and only if $\{0\}$ is a prime ideal.

11. (10 points) Prove the *freshman theorem* for groups: given a chain $A \leq B \leq G$ of normal subgroups of G,

$$(G/A)/(B/A) \cong G/B.$$

Hint: Start with a map $\phi\colon G/A \to G/B$, and make sure you define it.

12. (10 points) Make a list of all abelian groups of order 200, up to isomorphism. That is, each group should appear exactly once on your list.

13. (10 points) Prove that there are no simple groups of order $20 = 2^2 \cdot 5$. Clearly state what result(s) you are using.

14. (4 points) What was your favorite topic in this class? Specifically, what did you find the most interesting, and why?