Math 4120, Midterm 1. March 2, 2022

Write your answers for these problems *directly on this paper*. You should be able to fit them in the space given.

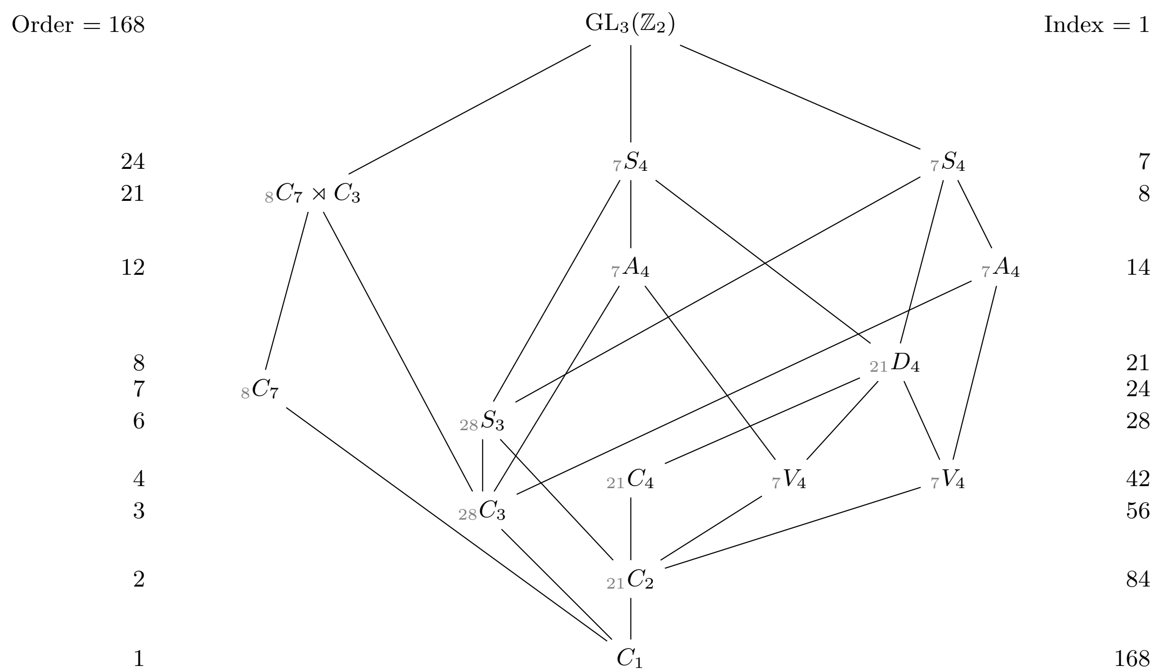
1. (10 points) Let G be a group with a subgroup $H = \langle b, c \rangle$. *Be as specific as possible with your answers.*

   (a) If $a \in H$, then what is $\langle a, b, c \rangle$?

   (b) If $a \notin H$ and $[G : H] = 2$, then what is $\langle a, b, c \rangle$?

   (c) If $a \notin H$, and $|G| = 48$ and $|H| = 6$, what are the possible orders of the subgroup $\langle a, b, c \rangle$?

2. (12 points) Answer the following questions about the second smallest nonabelian simple group, $G = \mathrm{GL}_3(\mathbb{Z}_2)$, whose reduced subgroup lattice is shown below. *Each justification should only be 1 senetence.*



   (a) Which subgroups of G are normal?

   (b) Consider an element $x \in G$ of order $|x| = 3$, and let $H = \langle x \rangle$. What is the normalizer $N_G(H)$ isomorphic to? Explain how you know this.

   (c) Circle the *fully unnormal* subgroups. How can you identify them?

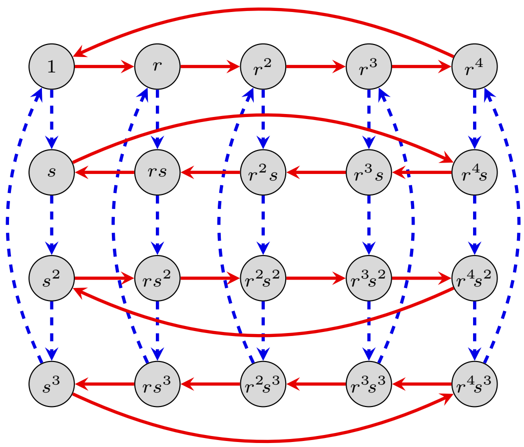
   (d) Box the *moderately unnormal* subgroups. How can you identify them?

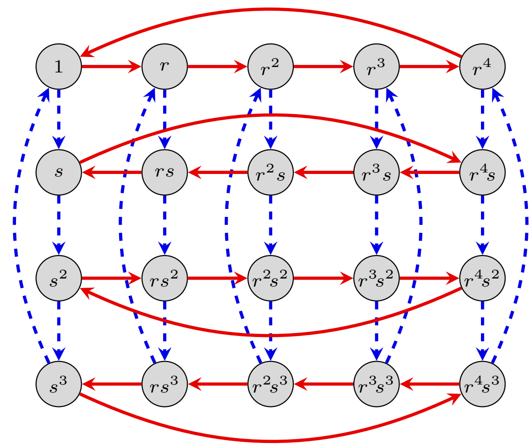
   (e) Explain why the *center*, $Z(G)$, cannot be equal to G.

   (f) Which subgroup is the center? Justify your answer.

3. (10 points) Make a list of all abelian groups of order $48 = 2^4 \cdot 3$. That is, every abelian group of order 48 should be isomorphic to precisely <u>one</u> group on your list. Write, e.g., $C_2^2 := C_2 \times C_2$ for short.

4. (24 points) Consider the Cayley diagram of the group $G = \langle r, s\rangle$ shown below, twice.





(a) Write a presentation for this group.

(b) Find all left cosets of $H = \langle r\rangle$, and then find all right cosets. Write them as subsets, or you can describe them in words, if applicable (e.g., the columns, the rows, etc.).

(c) Find all left cosets of $K = \langle s\rangle$, and then find all right cosets. Write them as subsets, or you can describe them in words, if applicable.

(d) Is H normal, moderately unnormal, or fully unnormal?

(e) Is K normal, moderately unnormal, or fully unnormal?

(f) Find the normalizers of H and K. Write them by generator(s), and say what familiar group each is isomorphic to.

(g) Find all conjugate subgroups to H and K. Write each group by generator(s).

(h) What is the order of the element rs^2?

(i) There are three nonabelian groups of order 20: D_{10}, Dic_{10}, and $\mathrm{AGL}_1(\mathbb{Z}_5)$. Which one is this? Justify your answer.

5. (10 points) Draw the *cycle diagram* (not the Cayley diagram) of the group

$$\mathbb{Z}_5 \times \mathbb{Z}_2 = \big\{(a,b) \mid a \in \mathbb{Z}_5,\ b \in \mathbb{Z}_2\big\}.$$

and then construct the subgroup lattice. Find two *minimal* generating sets of different sizes. Write your elements as ordered pairs (a,b), or as length-2 strings ab.

6. (16 points) Give an example of each of the following. *No justification needed.*

   (a) Two minimal generating sets of S_5 of different sizes.

   (b) A nonabelian group such that every subgroup is normal.

   (c) An element in S_5 of order 6. Use cycle notation.

   (d) An element in A_5 of order 2. Use cycle notation.

   (e) An infinite *noncyclic* abelian group.

   (f) A group G of order 16 such that $g^2 = e$ for all $g \in G$.

   (g) Two nonisomorphic subgroups of $D_4 = \langle r, f \rangle$ of the same order. (Write by generator(s)).

   (h) A subgroup $H \leq G$ and element $x \in G$ for which $xH = Hx$ holds setwise, despite $xh = hx$ not holding for all individual elements $h \in H$.

7. (10 points) Recall that the *center* of G is the set of elements that commute with everything:

$$Z(G) := \big\{z \in G \mid gz = zg,\ \text{for all } g \in G\big\}.$$

Use the standard (three-step) subgroup test to show that $Z(G)$ is a subgroup. Then show it is normal.

8. (8 points) Suppose that $N \leq G$ is a subgroup of index $[G : N] = 2$. Show that N is normal.