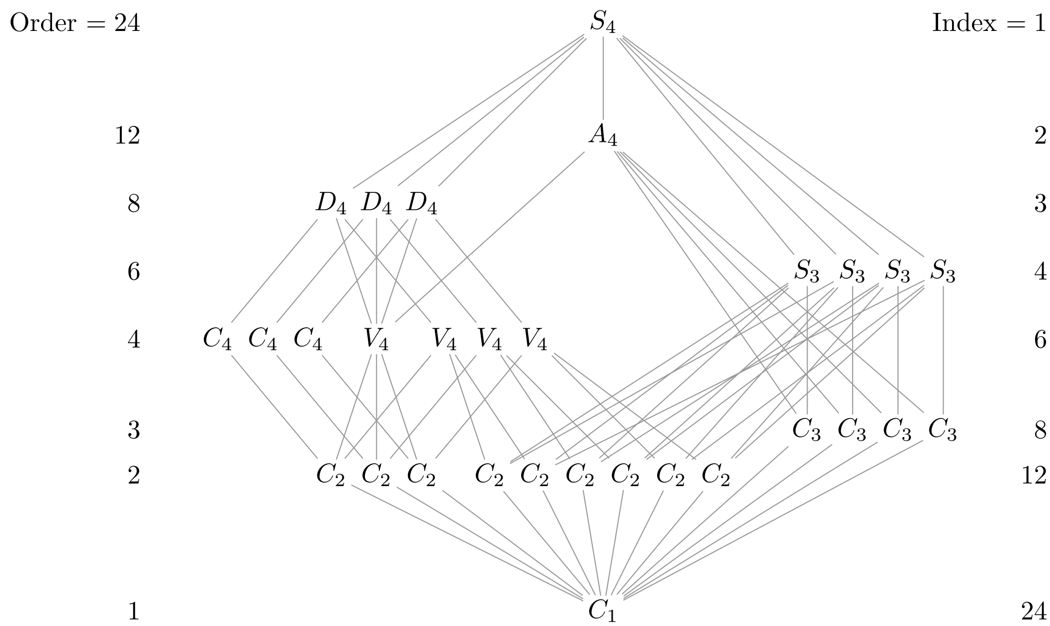
Math 4120, Midterm 2. April 20, 2022

Write your answers for these problems *directly on this paper*. You should be able to fit them in the space given.

1. (20 points) Answer the following questions about $G = S_4$, whose subgroup lattice is shown below.



(a) Circle each normal subgroup of S_4 on the lattice.

(b) Find the *commutator subgroup* G', and the *abelianization*, G/G'.

(c) Find the 2nd and 3rd commutator subgroups, $G'' := (G')'$ and $G''' := (G'')'$.

(d) Put a * next to the subgroup $H = \langle (12)(34), (13)(24) \rangle$ in the lattice.

(e) What is the quotient of S_4/H isomorphic to? How do you know?

(f) Write S_4 as a semidirect product of two proper subgroups, in as many distinct ways as possible.

(g) What is the *center* of S_4 isomorphic to, and why?

(h) For the remainder of this problem, suppose G acts on its subgroups by conjugation. How many orbits are there? How many fixed points?

(i) How many of the 30 subgroups do *not* arise as the stabilizer of another subgroup? Justify your answer for possible partial credit (in case you miscount or miss one, as is easy to do).

(j) What is the kernel of this action, and why?

2. (20 points) Complete the following statements, using formal mathematical language and/or notation. Make sure to correctly use, e.g., "for all" ($\forall$) and "there exists" ($\exists$) where appropriate.

   (a) A *group action* ϕ of G on a set S is (give a formal definition) ...

   (b) If G acts on S, then the *orbit* of the element $s \in S$ is the set:

   $$\mathrm{orb}(s) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\}.$$

   In particular, it is a subset of (the group G) (the set S) (neither). [$\longleftarrow$ circle one of these]

   (c) If G acts on S, then the *stabilizer* of an element $s \in S$ is the set:

   $$\mathrm{stab}(s) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\}.$$

   In particular, it is a subset of (the group G) (the set S) (neither).

   (d) If G acts on S, then the *fixed point set* of an element $g \in G$ is the set:

   $$\mathrm{fix}(g) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\}.$$

   In particular, it is a subset of (the group G) (the set S) (neither).

   (e) If G acts on S, then the *fixed points* of the action is the set:

   $$\mathrm{Fix}(\phi) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad \Big\} = \bigcap \qquad\qquad .$$

   In particular, it is a subset of (the group G) (the set S) (neither). Also, write it as an intersection.

   (f) If G acts on S, then the *kernel* of the action is the set:

   $$\mathrm{Ker}(\phi) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad \Big\} = \bigcap \qquad\qquad .$$

   In particular, it is a subset of (the group G) (the set S) (neither). Also, write it as an intersection.

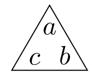
3. (8 points) Suppose G acts on S. Show that the stabilizer of any element $s \in S$ is a subgroup.

4. (10 points) Show that $H \cong xHx^{-1}$ for any $x \in G$. [*Hint*: Define a map and show it is a bijective homomorphism.]

5. (18 points) Let H, N be subgroups of G with $N \trianglelefteq G$.

   (a) Show that $H/(H \cap N) \cong HN/N$. (You may assume that $(H \cap N) \trianglelefteq H$ and $N \trianglelefteq HN$.)

   (b) The assumptions in this problem can actually be slightly weakened, and the same result will hold. Describe how, and loosely justify your answer (no need for a full proof).

6. (16 points) Let S be the set of $2^3 = 8$ "binary triangles:" $S = \Big\{$ 

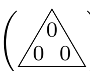
$: \ a, b, c \in \{0, 1\} \Big\}$.

Consider the action of $G = D_3$ on S, where

$\phi(r)$ = rotates each triangle 120° counterclockwise, $\phi(f)$ = reflects each triangle about its vertical axis.

(a) Draw the *action diagram*.

(b) Find the following:

- stab (

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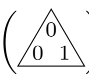
- stab (

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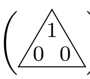
- stab (

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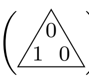
- stab (

) =
- $\text{fix}(f) =$
- $\text{fix}(rf) =$
- $\text{fix}(r) =$
- $\text{fix}(1) =$
- Average size of $\text{fix}(g)$, where $g \in D_3 =$
- $\text{Fix}(\phi) =$
- $\text{Ker}(\phi) =$

(c) (8 points) Suppose a group G of order 35 acts on a size-9 set S. Show there must be a fixed point. What can you say about the possible number of fixed points?

(d) (Extra credit, 2 points) What did the two little creatures at the top of this exam represent in the 3blue1brown videos that we watched?