Math 4130, Midterm 1. March 1, 2023

Write your answers for these problems *directly on this paper*. You should be able to fit them in the space given. If not, there is a blank page at the end of this exam.

1. (8 points) Finish the following formal mathematical definitions, where R is an integral domain.

   (a) An element $p \in R$ is *prime* if ...

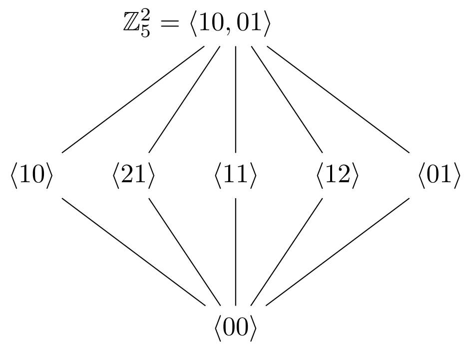
   (b) An element $m \in R$ is *irreducible* if ...

   Give an example of an element m in a ring R that is irreducible, but *not* prime. Justify your answer, and find an ideal for which $(m) \subsetneq I \subsetneq R$.

2. (8 points) Let R be a commutative ring, with a subring S and ideal I. Show that the subring $S \cap I$ of S is also an ideal of S. Sketch the portion of the subring lattice of R, that includes R, S, I, $S + I$, and $S \cap I$.

3. (7 points) Show that if an ideal I of R contains a unit, then $I = R$.

4. (7 points) The subgroup lattice for the ring $R = \mathbb{Z}_5^2$ is shown below.



(a) For each of the seven subgroups shown, do the following:

- If it is an ideal, then box it.
- If it is a subring but not an ideal, then circle it.
- If it is a subgroup but not a subring, then do nothing.

(b) Determine which the seven subgroups each of the following ideals is equal to:

i. $(10) =$ ii. $(11) =$ iii. $(12) =$

Recall that $(a) = \{ar \mid r \in R\}$, the "smallest ideal that contains a."

5. (7 points) Let R be a commutative ring. Prove that every maximal ideal M of R is prime. [*Hint*: consider R/M.]

6. (10 points) Finish the following sentences, so they are *formal* mathematical definitions. Make sure you use terminology like "for all", where appropriate.

   (a) A *homomorphism* ϕ from a ring R to S is...

   (b) A *left ideal* I of R is...

   (c) An element $u \in R$ is a *unit* if ...

   (d) A ring R is an *integral domain* if ...

   (e) A ring R is *Noetherian* if ...

7. (16 points) Fill in the following blanks.

   1. An example of a nontrivial ring *without* unity is ____________________.
   2. The field of fractions of the ring $R = \mathbb{R}$ of real numbers is ____________________.
   3. In $R = \mathbb{Z}_6[x]$, the element ________$\neq 1$ is a unit, ________$\neq 0$ is a zero divisor, and ________ is neither.
   4. An example of a non-commutative division ring is ____________________.
   5. The quadratic integer ring $R = \mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is one of 4 that is a ____________ but not a ____________.
   6. Two ideals I and J are *co-prime* if ____________________.
   7. An example of a non-maximal prime ideal is $I =$________________ in the ring $R =$________________.
   8. An example of a subring of $\mathbb{Z}[x]$ that is *not* an ideal is ____________________.
   9. An example of a non-finitely generated ideal of $\mathbb{Z}[x_1, x_2, \dots]$ is ____________________.
   10. By Zorn's lemma, every ideal $I \subsetneq R$ is contained in a ____________________.
   11. An ideal $M \subsetneq R$ is *maximal if* $M \subseteq I \subseteq R$ implies ____________________.
   12. Every simple commutative ring is ____________________.
   13. The quotient ring $\mathbb{Z}_3[x]/(x^4+1)$ is the finite field with ____________________ elements.

8. (7 points) Let $\phi\colon R \to S$ be a homomorphism and let $I = \mathrm{Ker}(\phi)$. Show that I is a left ideal. You may assume that it is an additive subgroup.

9. (7 points) The *diamond theorem* says that $(S+I)/I \cong S/(S\cap I)$ for a subring S and ideal I. In proving the diamond theorem for groups, the map

$$\phi\colon S \longrightarrow (S+I)/I, \qquad \phi(s) = s+I$$

was shown to be a surjective *group* homomorphism with $\mathrm{Ker}(\phi) = S \cap I$. You may assume this. Carry out the remaining details to establish the diamond theorem for rings.

10. (20 points) One of the themes in this part of the class is that "*questions about divisbility are best phrased in the language of ideals*." Write several paragraphs about this, where your target reader is someone who just finished Math 4120 (Algebra 1), and knows what rings and ideals are. Explain *how* divisibility is encoded by ideals, and *why* it is so clean. What are the interpretations of irreducible and prime elements in terms of ideals, and how are they similar/different? Discuss what a PID is, and give examples of "nice results" that occur in PIDs, and *why* one would expect these rings to exhibit such nice behavior. Finally, briefly discuss the difference between UFDs, PIDs, Euclidean domains, and fields, with examples and non-examples to illustrate.

11. (4 points) What was your favorite part about ring theory, and why? Specifically, what topic(s) did you find the most interesting, and what about them did you most enjoy?