1. Let R be a commutative ring with 1.

   (a) Show that the following two conditions are equivalent:

   (i) the non-units of R form an ideal,

   (ii) R has a unique maximal ideal.

   (b) Characterize the units of the following ring

   $$S = \big\{a/b \mid a, b \in \mathbb{Z},\ \gcd(a, b) = 1,\ b \text{ is odd}\big\},$$

   and show that the nonunits form a unique maximal ideal.

2. Let R be a commutative ring with 1, and $D \subseteq R$ a multiplicatively closed subset containing no zero divisors. Consider the following set and equivalence relation $\sim$:

   $$R \times D = \big\{(r, d) \mid r \in R,\ d \in D\big\}, \qquad (r_1, d_1) \sim (r_2, d_2) \ \Leftrightarrow\ r_1 d_2 = r_2 d_1.$$

   (a) Show that $\sim$ is an equivalence relation.

   (b) Let r/d denote the equivalence class containing (r, d), and the set of equivalence classes by $D^{-1}R$. Define addition and subtraction as follows:

   $$\frac{r_1}{d_1} + \frac{r_2}{d_2} := \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \qquad \text{and} \qquad \frac{r_1}{d_1} \times \frac{r_2}{d_2} := \frac{r_1 r_2}{d_1 d_2}.$$

   Show that these operations are well-defined.

   (c) Show that the additive and multiplictive identities are $0/d$ and d/d, for any $d \in D$, and that the multiplicative inverse of r/d, if it exists, is $(r/d)^{-1} = d/r$.

   (d) If $d \in D$, show that $\{rd/d \mid r \in R\}$ is a subring of $D^{-1}R$ and that

   $$R \longrightarrow D^{-1}R, \qquad r \longmapsto rd/d$$

   is a injective homomorphism, therefy identifying R with a subring of $D^{-1}R$.

   (e) Under this identification, show that every $d \in D$ gets mapped to a unit in $D^{-1}R$.

   (f) Show how the ring S from the previous problem arises as ring $D^{-1}R$ of fractions, for some D and R.

3. Let D be a multiplicatively closed subset of a commutative ring R, and $f\colon R \hookrightarrow S$ an embedding to a commutative ring with 1, such that $f(d)$ is a unit, for every $d \in D$. Show that there is a unique ring homomorphism $h\colon D^{-1}R \hookrightarrow S$ such that $h \circ \iota = f$.