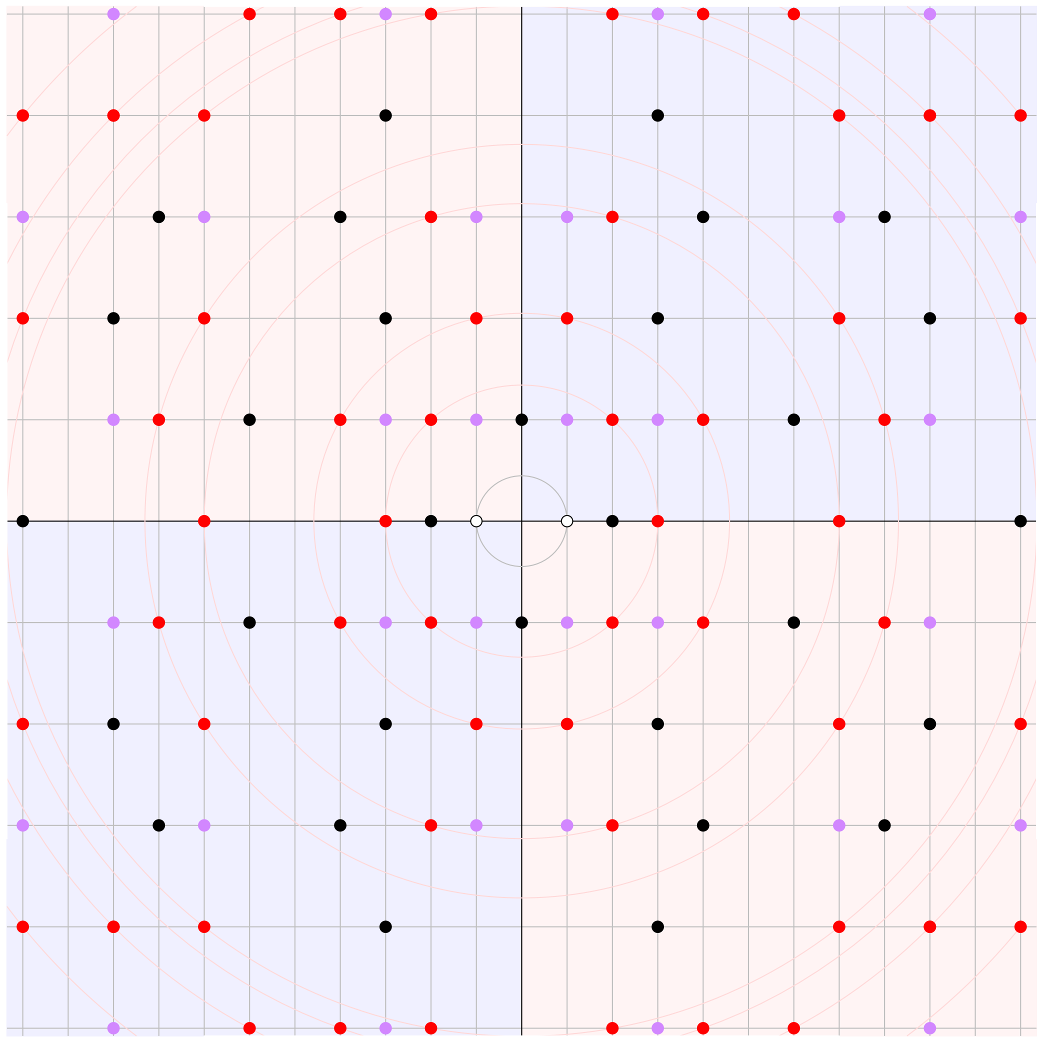
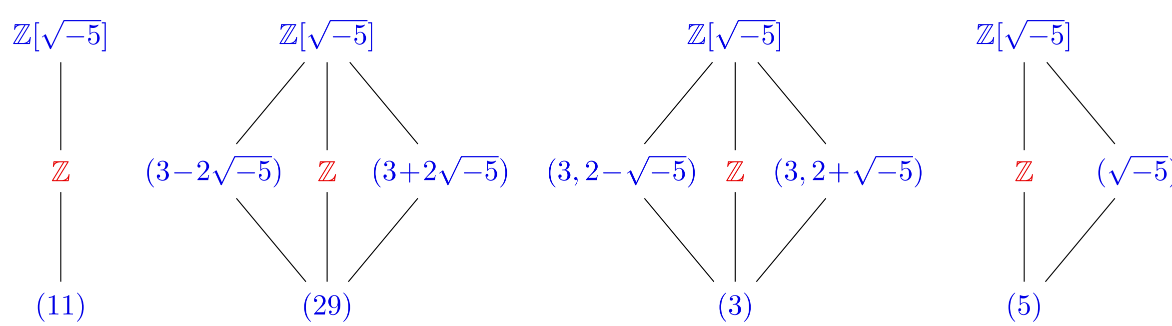
1. A picture illustrating the quadratic integers $R_{-5} = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ as a subring of $\mathbb{C}$ is shown below, with the primes in black, and non-prime irreducibles in red.



   (a) Create an analogous picture for the ring $R_{-6} = \mathbb{Z}[\sqrt{-6}]$. First make a blank diagram with the norms of the quadratic integers labeled at each correponding lattice point.

   (b) Find primes $p \in \mathbb{Z}$ that in R_{-6} are inert, split (both reducible and irreducible), and ramified. Illustrate this with subring lattices. Examples for R_{-5} are shown below.



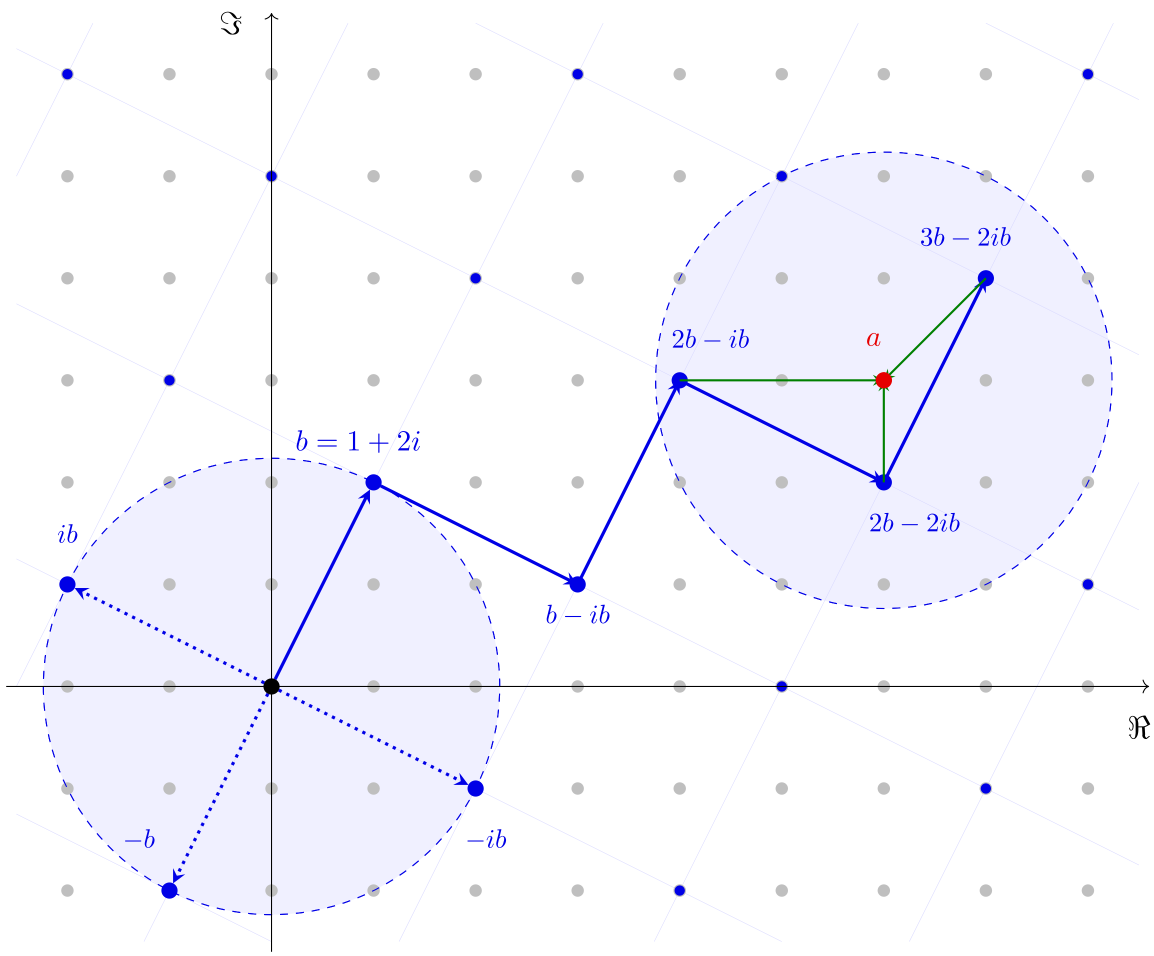
   (c) Give an elementary characterization of non-prime irreducibles in R_{-6}.

2. Prove the following basic facts about principal ideal domains (PIDs).
   (a) The following three conditions are equivalent for nonzero $a, b \in R$:
      (i) a and b are associates (that is, $a \mid b$ and $b \mid a$),
      (ii) $a = bu$ for some unit $u \in R$,
      (iii) $(a) = (b)$.
   (b) The following three conditions are equivalent for an element $a \in R$:
      (i) a is irreducible
      (ii) the ideal (a) is maximal
      (iii) the ideal (a) is prime
   (c) Any two nonzero elements $a, b \in R$ have a unique LCM, up to associates, which is a generator m of $(a) \cap (b)$.

3. If the quadratic integer ring R_m is a Euclidean domain, then for every nonzero $a, b \in R$, the division algorithm can be used to find $r, q \in R$ such that

$$a = bq + r, \qquad 0 \le N(r) < N(b).$$

   (a) The following visual shows all three ways to write $a = bq + r$ in the Gaussian integers, for $a = 6 + 3i$ and $b = 1 + 2i$. Carry out the division algorithm for $a = 9 + 8i$ and $b = 3 + i$, and create an analogous picture to illustrate it.



(b) The ring $R_{-5} = \mathbb{Z}[\sqrt{-5}]$ is *not* a Euclidean domain, since the division algorithm fails for $a = 5$ and $b = 2 + \sqrt{-5}$, as demonstrated by the following visual. Find an a and b in $R_{-6} = \mathbb{Z}[\sqrt{-6}]$ that confirms that it too is not Euclidean, and construct an analogous visual.