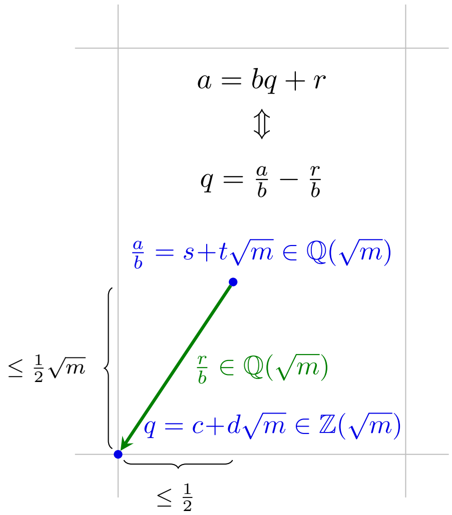
1. The "rectangular" quadratic integer rings $R_m = \mathbb{Z}[\sqrt{m}]$ are norm-Euclidean for $m = -2, -1, 2, 3$. The proof involves constructing the quotient q, as the nearest element to a/b, as shown below:



For each $m = -2, -1, 2, 3$,

$$-1 < N(\tfrac{r}{b}) = \underbrace{(c-s)^2}_{\leq \frac{1}{4}} - m\underbrace{(d-t)^2}_{\leq \frac{1}{4}} < 1.$$

Prove the same result for the "triangular" quadratic integer ring $R_m = \mathbb{Z}\big[\frac{1+\sqrt{m}}{2}\big]$, for $m = -3$, -7, and -11. Include an analogous picture illustrating the idea. [*Hint*: Mimic the proof of the rectangular cases, but note that to reach nearest quadratic integer, one needs to travel at most a horizontal distance of $1/2$ and a vertical distance of $\sqrt{m}/2$.]

2. Find a solution to each of the following systems of equations, as guaranteed by the Sunzi remainder theorem, and show your work.

   (a) In the ring $\mathbb{Z}$ of integers:

   $$x \equiv 1 \pmod 8, \qquad x \equiv 3 \pmod 7, \qquad x \equiv 9 \pmod{11}$$

   (b) In the ring $R_{-1} = \mathbb{Z}[i]$ of Gaussian integers:

   $$x \equiv i \pmod{i+1}, \qquad x \equiv 1 \pmod{2-i}, \qquad x \equiv 1+i \pmod{3+4i}$$

   (c) In $F[x]$, where F is a field in which $1 + 1 \neq 0$:

   $$f(x) \equiv 1 \pmod{x-1}, \qquad f(x) \equiv x \pmod{x^2+1}, \qquad f(x) \equiv x^3 \pmod{x+1}$$

3. An *idempotent* is an element $e \in R$ satisfying $e^2 = e$. Two nonzero idempotents e_1, e_2 are an *orthogonal pair* if $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.

   (a) Show that the following are equivalent:
      (i) R contains an idempotent different from 0 and 1.
      (ii) R contains an orthogonal pair of idempotents.
      (iii) $R \cong R_1 \times R_2$ for some rings R_1 and R_2.

   (b) Give an example of a non-orthogonal pair of distinct idemponents.

   (c) Find all idempotents in the ring $\mathbb{Z}/20\mathbb{Z} \cong \mathbb{Z}_5 \times \mathbb{Z}_4$.