1. (a) Show that $f(x) = x^4 + 10x^3 - 16x^2 + 6x + 2$ is irreducible over $\mathbb{Q}$.
   (b) Show that $f(x) = x^p + p - 1$ is irreducible over $\mathbb{Q}$, for any prime p. [Hint: $f(x)$ is irreducible iff $f(x+1)$ is.]
   (c) Show that
   $$\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}, \qquad \text{and} \qquad \mathbb{C}[x]/(x^2+1) \cong \mathbb{C} \times \mathbb{C}.$$
   Clearly state any results that you use.

2. Consider the following rings R_i, for $i = 1, \dots, 6$, which are additionally $\mathbb{C}$-vector spaces:
   $$R_1 = \mathbb{C}[x]/(x^3 - 1)$$
   $$R_2 = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$$
   $$R_3 = \text{the ring of upper triangular } 2 \times 2 \text{ matrices over } \mathbb{C}$$
   $$R_4 = \mathbb{C}[x]/(x-1) \times \mathbb{C}[x]/(x+i) \times \mathbb{C}[x]/(x-i)$$
   $$R_5 = \mathbb{C}[x]/(x^2+1) \times \mathbb{C}[x]/(x-1)$$
   $$R_6 = \mathbb{C}[x]/(x+1)^2 \times \mathbb{C}[x]/(x-1).$$
   (a) Compute the dimension of each R_i as a $\mathbb{C}$-vector space by giving an explicit basis.
   (b) Partition the rings $R_1, \dots, R_6$ into isomorphism classes, with justification.

3. Let R be a commutative ring with identity and F a field.
   (a) Show that if R is a PID then any nonzero prime ideal $P \subseteq R$ is maximal.
   (b) Show that there is a bijective correspondence between maximal ideals of $F[x]$ and monic irreducible polynomials in $F[x]$.
   (c) Show that if $M \subsetneq \mathbb{Z}[x]$ is a maximal ideal, then $M \bigcap \mathbb{Z} = (p)$ for some prime $p \neq 0$.
   (d) Show that there is a bijective correspondence between maximal ideals of $\mathbb{Z}[x]$ that contain p and monic irreducible polynomials in $\mathbb{Z}_p[x]$.
   (e) Characterize all maximal ideals of $\mathbb{Z}[x]$.

4. Let I be an ideal of $R[x]$, where R is an integral domain. Define
   $$I(m) = \big\{ a_m \mid f(x) = a_m x^m + \cdots + a_1 x + a_0 \in I \big\} \cup \{0\} \trianglelefteq R.$$
   (a) Show that $I(m)$ is an ideal of R.
   (b) Show that $I(m) \subseteq I(m+1)$.
   (c) Show that if $I \subseteq J$, then $I(m) \subseteq J(m)$.

   Do any of these three results use the assumption that R has unity?