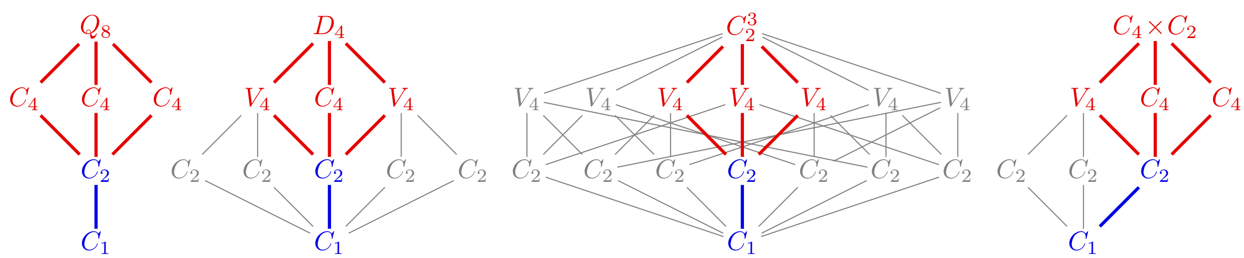
1. Consider the commutative diagram of exact sequences shown at left.

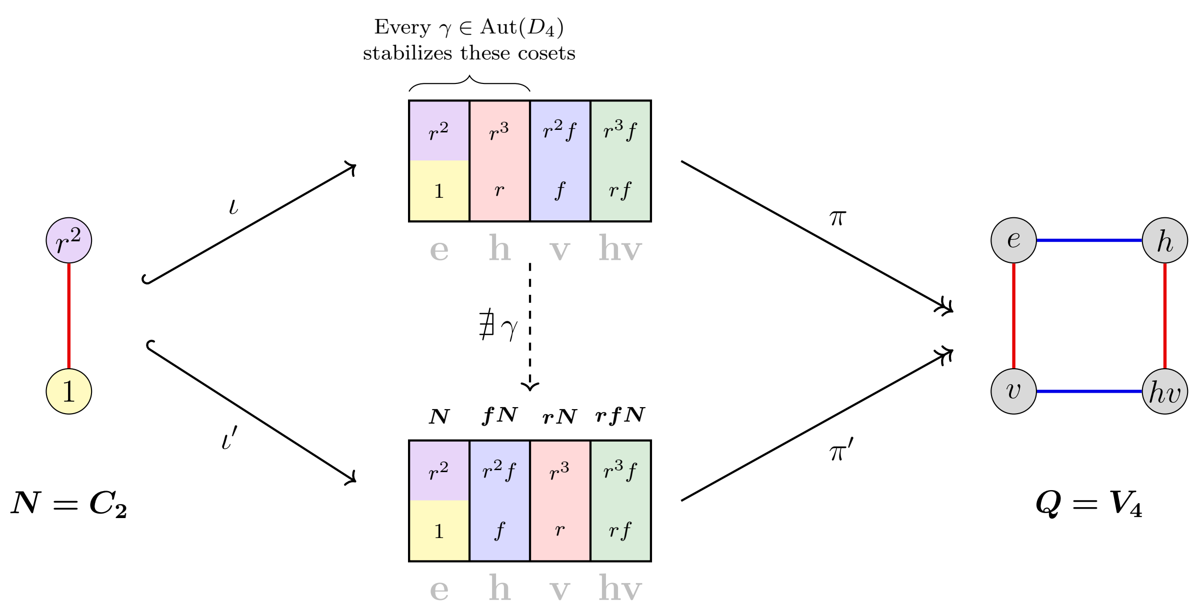
$$\begin{tikzcd} 1 \arrow[r] & N_1 \arrow[r, "\iota_1"] \arrow[d, "\nu"] & G_1 \arrow[r, "\pi_1"] \arrow[d, "\gamma"] & Q_1 \arrow[r] \arrow[d, "\kappa"] & 1 \\ 1 \arrow[r] & N_2 \arrow[r, "\iota_2"] & G_2 \arrow[r, "\pi_2"] & Q_2 \arrow[r] & 1 \end{tikzcd} \qquad \begin{tikzcd} & & G \arrow[dd, dashed, "\gamma"] \arrow[dr, "\pi"] & & \\ 1 \arrow[r] & N \arrow[ur, hook, "\iota"] \arrow[dr, hook, "\iota'"'] & & Q \arrow[r] & 1 \\ & & G \arrow[ur, "\pi'"'] & & \end{tikzcd}$$

(a) Prove that if ν and κ are isomorphisms, then γ is as well.

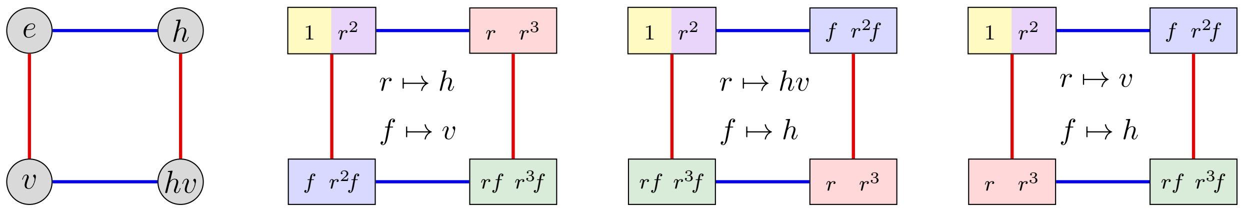
(b) Two extentions $N_i \overset{\iota_i}{\hookrightarrow} G_i \overset{\pi_i}{\twoheadrightarrow} Q_i$ are said to be *equivalent* if they are related via a commutative diagram, like the one above (left). Two extensions of Q by N are equivalent if they are related by a commutative diagram like the one above (right). Up to *isomorphism*, there are four extensions of $Q = V_4$ by $N = C_2$, as shown below.



However, up to equivalence, there are more than four. For example, finding all extesnions $C_2 \overset{\iota}{\hookrightarrow} D_4 \overset{\pi}{\twoheadrightarrow} V_4$ amounts to finding all $\gamma \in \mathrm{Aut}(D_4)$ that makes the diagram commute. Since each γ fixes the cosets $\{1, r^2\}$ and $\{r, r^3\}$, the following diagram illustrates two quotients that cannot be equivalent.



Each of the three choices of the image $\pi(r) \in \{h, v, hv\}$ characterizes an extension $C_2 \overset{\iota}{\hookrightarrow} D_4 \overset{\pi}{\to} V_4$. Examples of these are shown below.



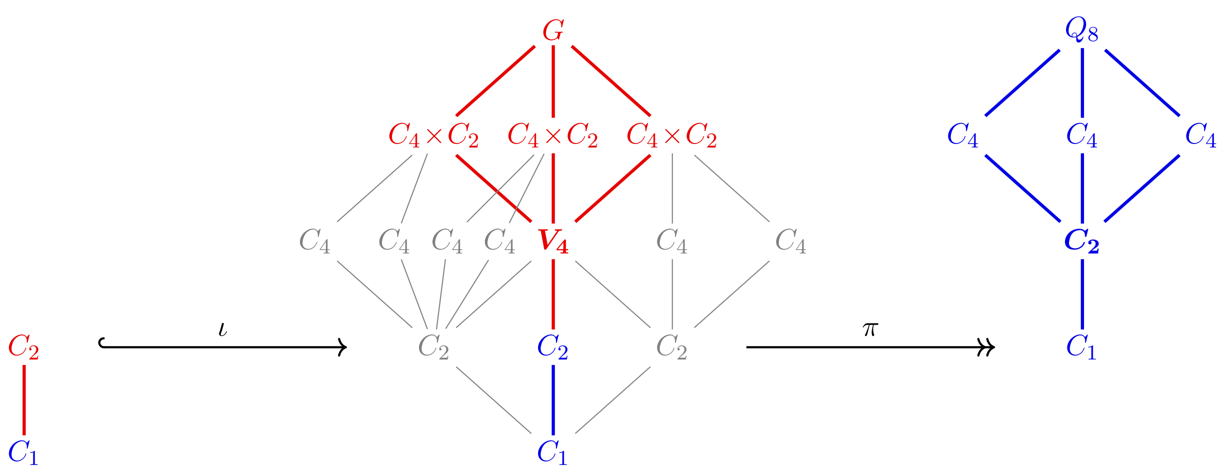
Carry out the previous steps, including the visuals, to classify the extensions $C_2 \overset{\iota}{\hookrightarrow} C_4 \times C_2 \overset{\pi}{\to} V_4$, and then do the same for the extensions $C_2 \overset{\iota}{\hookrightarrow} Q_8 \overset{\pi}{\to} V_4$.

2. Consider a sequence of homomorphisms between groups:

$$1 \longrightarrow A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 1$$

This sequence is said to be *exact at* B if $\operatorname{Im}(f) = \operatorname{Ker}(g)$. We can define exactness at A and C analogously. The entire sequence is exact if it is exact at A, B, and C. Show that this implies that f is injective and g is surjective.

3. The exact sequence below demonstrates the construction of a group G of order 16, as a central (and hence abelian), nonsplit extension of $Q = Q_8$ by $N = C_2$.



Though there are more up to equivalence, up to isomorphism of the factors, the group G can be constructed as an extension of Q by N in six distinct ways, where neither factor is trivial. Create an exact sequence with subgroup lattices, like the one above, for the other five. For each, decide whether it is split, abelian, and central. Note that it can be more than one of these, or none at all.