1. The splitting field of the polynomial $f(x) = x^4 - 2$ is

$$K = \mathbb{Q}(\sqrt[4]{2}, i) = \left\{a + b\sqrt[4]{2} + c\sqrt{2} + d\sqrt[4]{8} + ei + f\sqrt[4]{2}i + g\sqrt{2}i + h\sqrt[4]{8}i \mid a, \dots, h \in \mathbb{Q}\right\},$$

which has degree $[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}] = 8$ over $\mathbb{Q}$.

(a) There are eight non-trivial proper subfields of $\mathbb{Q}(\sqrt[4]{2}, i)$:

- Five of degree 4 over the rationals: $\mathbb{Q}(\sqrt[4]{2})$, $\mathbb{Q}(i\sqrt[4]{2})$, $\mathbb{Q}(\sqrt{2}, i)$, $\mathbb{Q}((1+i)\sqrt[4]{2})$, and $\mathbb{Q}((1-i)\sqrt[4]{2})$.
- Three of degree 2 over the rationals: $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(i)$, and $\mathbb{Q}(i\sqrt{2})$.

For each subfield F, find a polynomial $f(x)$ of minimal degree that has a root r such that $F = \mathbb{Q}(r)$. Then determine how many of the roots of $f(x)$ lie in K.

(b) Arrange the 10 subfields in a lattice, with $\mathbb{Q}(\sqrt[4]{2}, i)$ on top, and $\mathbb{Q}$ on the bottom.

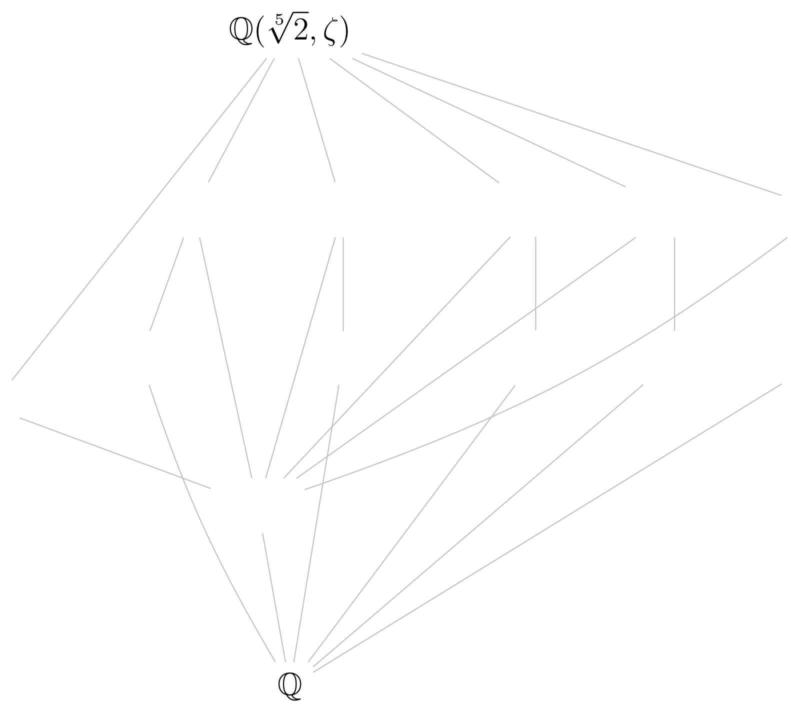
(c) The field $\mathbb{Q}(\sqrt[4]{2}, i)$ can be generated by just $\sqrt[4]{2} + i$, which is called a *primitive element*. For each of the ten subfields $F \subseteq \mathbb{Q}(\sqrt[4]{2}, i) = \mathbb{Q}(\sqrt[4]{2} + i)$, find a primitive element α over F, and its minimal polynomial. The degree of the polynomial will be equal to the degree of the extension $[F : \mathbb{Q}]$.

2. Consider the polynomial $f(x) = x^5 - 2$.

(a) Show that the splitting field of $f(x) = x^5 - 2$ is $K = \mathbb{Q}(\sqrt[5]{2}, \zeta)$, where $\zeta = e^{2\pi i/5}$ is a primitive 5^{th} root of unity.

(b) In the complex plane, sketch all five roots of $f(x)$, and all five fifth roots of unity.

(c) Find the subfields of K, and arrange them in a subfield lattice, whose structure is shown below.



3. Consider the function

$$\phi\colon \mathbb{Q}(\sqrt{2}) \longrightarrow \mathbb{Q}(\sqrt{2})\,, \qquad \phi(a+b\sqrt{2}) = a - b\sqrt{2}\,.$$

   Show that ϕ is a field automorphism.

4. Let α be a root of a polynomial $f(x)$ that irreducible over $\mathbb{Q}$, which means that it generates a maximal ideal in $\mathbb{Q}[x]$.

   (a) Show that

   $$\phi\colon \mathbb{Q}[x] \longrightarrow \mathbb{Q}(\alpha), \qquad \phi\colon f(x) \longmapsto f(\alpha)$$

   is a ring homomorphism.

   (b) Show that $\mathbb{Q}[x]/(f(x)) \cong \mathbb{Q}(a)$.

   (c) Show that if β is another root of $f(x)$, then there is a field isomorphism $\mathbb{Q}(\alpha) \to \mathbb{Q}(\beta)$ that fixes $\mathbb{Q}$, elementwise, and sends $\alpha \mapsto \beta$.