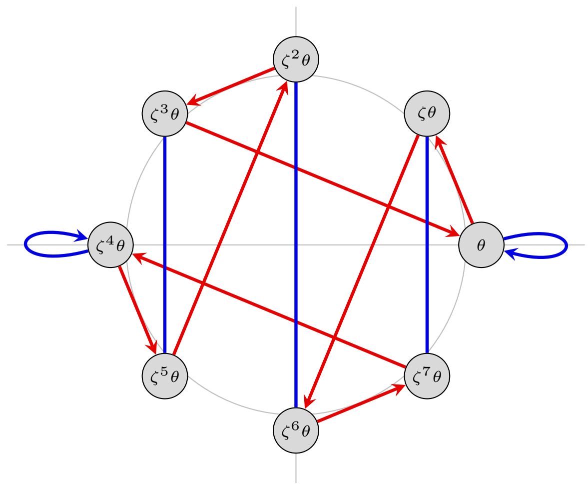
1. The splitting field of $f(x) = x^8 - 2$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[8]{2}, \zeta) = \mathbb{Q}(\sqrt[8]{2}, i)$, where $\zeta = e^{2\pi i/8} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$, a primitive $8^{\text{th}}$ root of unity. The Galois group is generated by the automorphisms

$$\begin{cases} \rho\colon \sqrt[8]{2} \longmapsto \zeta \sqrt[8]{2} \\ \rho\colon \quad \zeta \longmapsto \zeta \end{cases} \qquad \begin{cases} \sigma\colon \sqrt[8]{2} \longmapsto \sqrt[8]{2} \\ \sigma\colon \quad i \longmapsto -i\,, \end{cases}$$

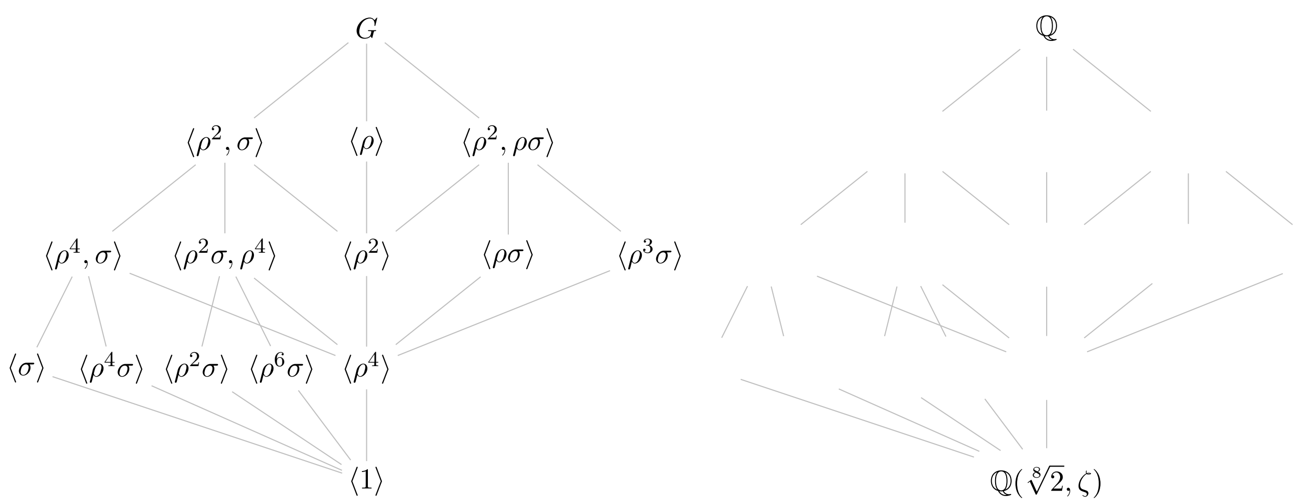
and is isomorphic to the semidihedral group

$$\mathrm{SD}_8 = \langle \rho, \sigma \mid \rho^8 = 1,\ \sigma^2 = 1,\ \sigma\rho\sigma = \rho^3 \rangle.$$

An action graph of $\mathrm{Gal}(x^8 - 2)$ acting on the roots is shown below, where $\theta = \sqrt[8]{2}$.




(a) For each subgroup $H \leq \mathrm{Gal}(x^8 - 2)$, find the largest subgroup of $\mathbb{Q}(\sqrt[8]{2}, i)$ fixed by $H$, and write it in the corresponding place on the subfield lattice on the right.




It is helpful to know that the proper subfields of $\mathbb{Q}(\sqrt[8]{2}, i)$ are: $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt[4]{2})$, $\mathbb{Q}(\sqrt[8]{2})$, $\mathbb{Q}(\sqrt{2}i)$, $\mathbb{Q}(\sqrt[4]{2}i)$, $\mathbb{Q}(\sqrt[8]{2}i)$, $\mathbb{Q}(\sqrt{2}, i)$, $\mathbb{Q}(\sqrt[4]{2}, i)$, $\mathbb{Q}((1+i)\sqrt[4]{2})$, $\mathbb{Q}((1-i)\sqrt[4]{2})$, $\mathbb{Q}(\zeta\sqrt[8]{2})$, $\mathbb{Q}(\zeta^3\sqrt[8]{2})$.

(b) Circle each subfield $E$ that is a normal extension of $\mathbb{Q}$, and find a polynomial whose splitting field over $\mathbb{Q}$ is $E$.

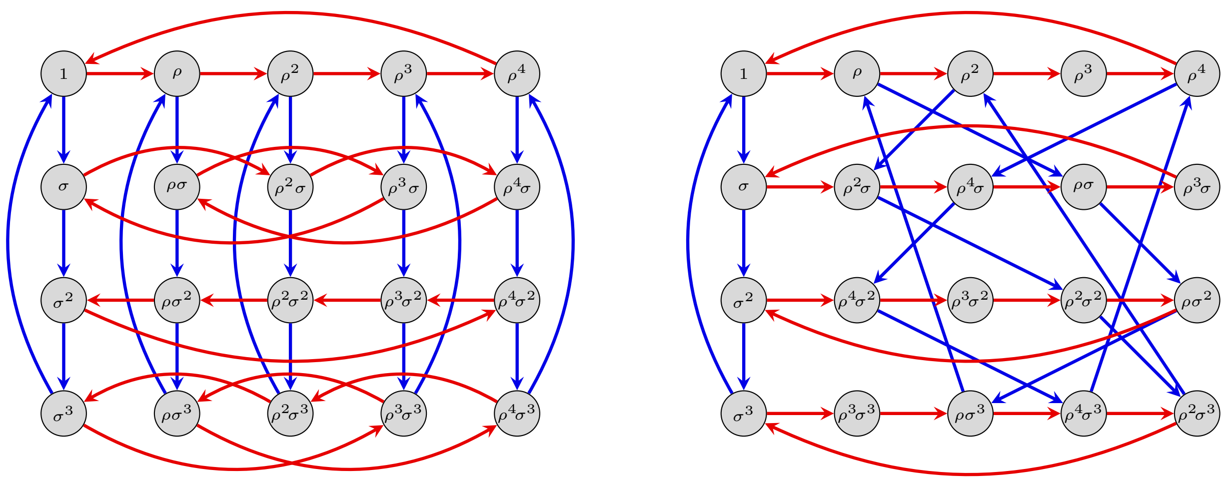
2. The Galois group of $f(x) = x^5 - 2$ is generated by the automorphisms

$$\begin{cases} \rho\colon \sqrt[5]{2} \longmapsto \zeta\sqrt[5]{2} \\ \rho\colon \quad \zeta \longmapsto \zeta \end{cases} \qquad \begin{cases} \sigma\colon \sqrt[5]{2} \longmapsto \sqrt[5]{2} \\ \sigma\colon \quad \zeta \longmapsto \zeta^2, \end{cases}$$

and is isomorphic to the affine general linear group

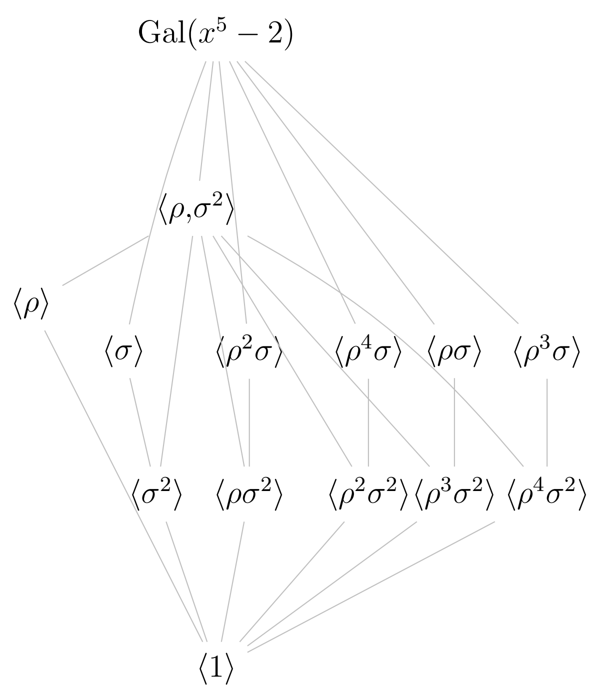
$$\mathrm{AGL}_2(\mathbb{Z}_5) = \langle \rho, \sigma \mid \rho^5 = 1,\ \sigma^4 = 1,\ \rho\sigma = \sigma\rho^3 \rangle.$$

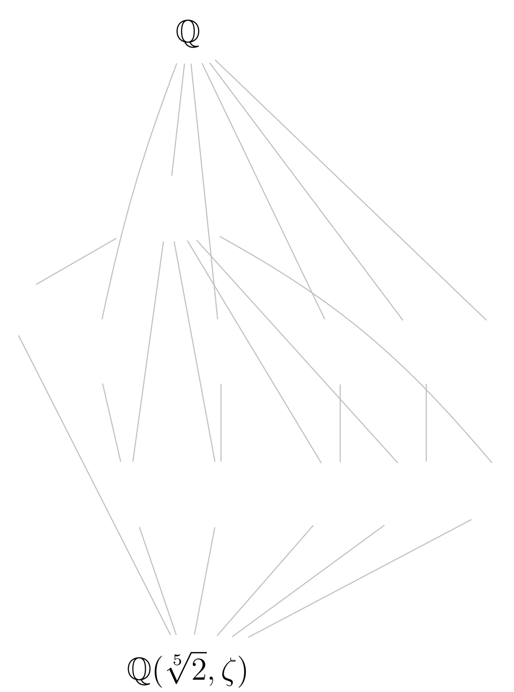
Two Cayley graphs are shown below.



(a) Draw the action graph of $G = \mathrm{Gal}(x^5 - 2)$ acting on the set $S$ of roots of $x^5 - 2$, like what was done in the previous problem.

(b) For each subgroup $H \leq \mathrm{Gal}(x^5 - 2)$, find the largest subgroup of $\mathbb{Q}(\sqrt[5]{2}, \zeta)$ fixed by $H$, and write it in the corresponding place on the subfield lattice on the right.



3. Repeat the previous problem for the Galois group of $f(x) = x^6 - 2$, which is generated by the automorphisms

$$\begin{cases} \rho\colon \sqrt[6]{2} \longmapsto \zeta\sqrt[6]{2} \\ \rho\colon \quad \zeta \longmapsto \zeta \end{cases} \qquad\qquad \begin{cases} \sigma\colon \sqrt[6]{2} \longmapsto \sqrt[6]{2} \\ \sigma\colon \quad \zeta \longmapsto \overline{\zeta}\,, \end{cases}$$

and is isomorphic to the dihedral group

$$D_6 = \langle \rho, \sigma \mid \rho^6 = 1,\ \sigma^2 = 1,\ \rho\sigma = \sigma\rho^5 \rangle.$$

The subgroup and blank subfield lattice are shown below.