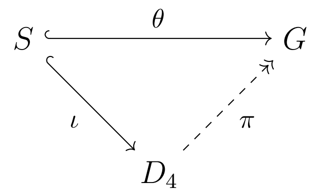
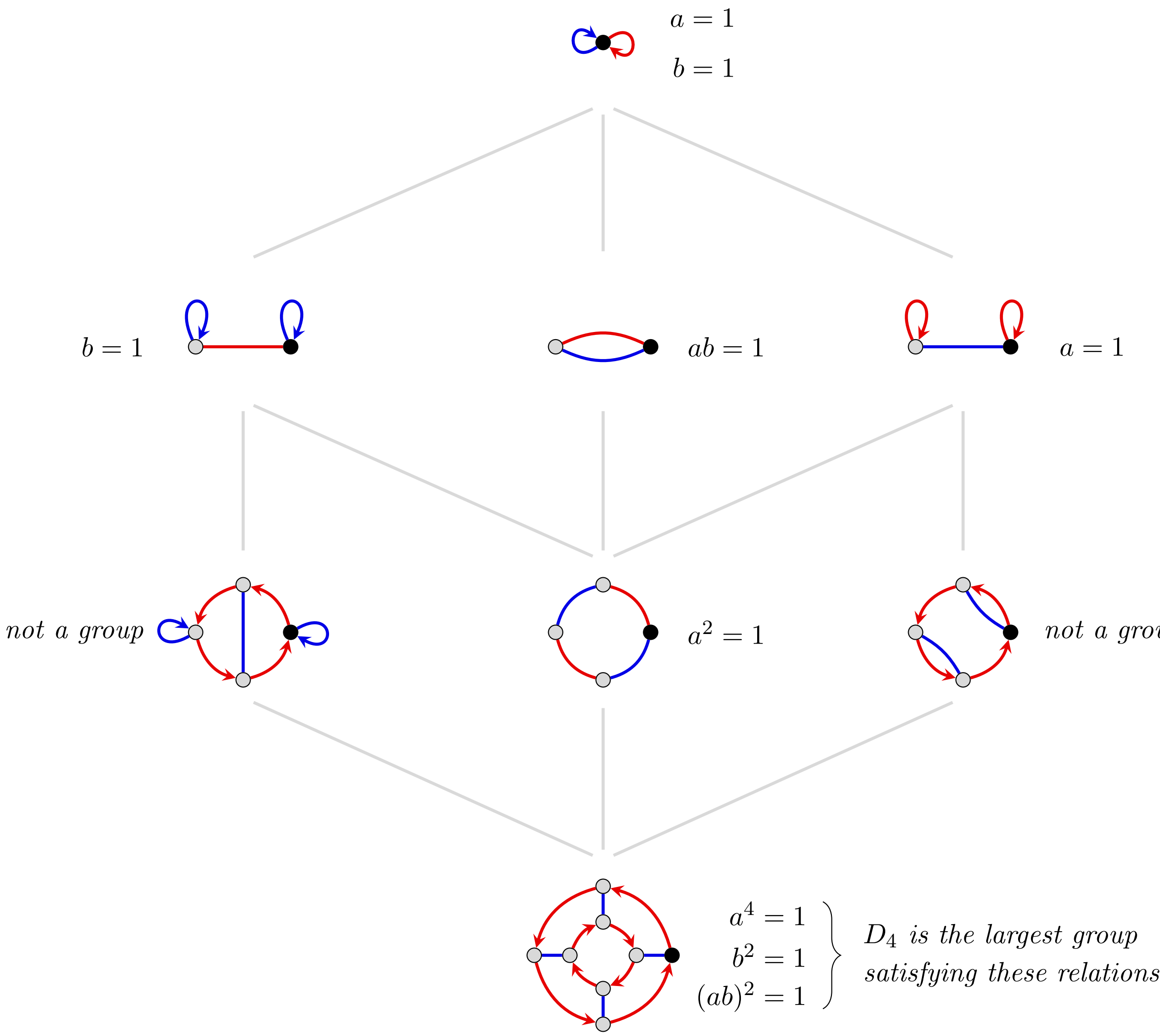
1. Every group $G = \langle a, b\rangle$ satisfying the relations $a^4 = 1$, $b^2 = 1$, and $(ab)^2 = 1$ is isomorphic to a quotient of D_4. Formally, this means that there is a unique homomorphism $D_4 \twoheadrightarrow G$ making the following diagram commute:



where ι and θ are the inclusion maps $a \mapsto a$ and $b \mapsto b$. The figure below shows all of the distinct ways to collapse the Cayley diagram for D_4 by right cosets of a subgroup, and the correponding relation(s) added, if the result is a group.

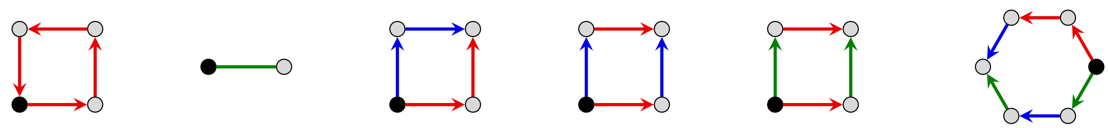


Create an analogous figure for $C_4 \rtimes C_4 = \langle a, b \mid a^4 = b^4 = abab^{-1} = 1\rangle$.

2. Consider the "mystery group" $M = \langle S_1 \mid R_1 \rangle$ defined by the following presentation.

$$M = \left\langle a, b, c \mid a^4,\ c^2,\ a^2b^{-2},\ aba^{-1}b^{-1},\ aca^{-1}c,\ a^2bc^{-1}b^{-1}c^{-1} \right\rangle.$$

The relators of this presentations describe the following motifs that a Cayley graph for $M = \langle a, b, c \rangle$ must have.



$a^4 = 1$ $\quad$ $c^2 = 1$ $\quad$ $a^2 = b^2$ $\quad$ $ab = ba$ $\quad$ $ac = ca$ $\quad$ $a^2b = cbc$

(a) Establish $|M| \leq 16$ by showing that every word in M can be written

$$a^i b^j c^k, \qquad i \in \{0,1,2,3\}, \quad j \in \{0,1\}, \quad k \in \{0,1\},$$

(b) Identify a "familiar group" $F = \langle S_2 \mid R_2 \rangle$ of order 16 whose generators satisfy these relations. That is, define a "relabing map" $\theta \colon S_1 \twoheadrightarrow S_2$ that extends to $\theta \colon R_1 \to R_2$.

(c) Describe why it follows that $M \cong F$.

3. Determine which group is described by each presentation, and prove that your answer is correct.

(a) $G = \langle a, b \mid a^2 = 1,\ b^3 = 1,\ ab = ba \rangle$

(b) $G = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ ab = ba^3 \rangle$

(c) $G = \langle a, b \mid a^4 = b^3 = 1,\ ab = ba^3 \rangle$

(d) $G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^3 = 1 \rangle$.