Chapter 2: Examples of groups

Matthew Macauley

Department of Mathematical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 4120 & 4130, Visual Algebra

Families of groups

In the previous chapter, we encoutered groups meant to appeal to intuition and motivate key concepts. In this chapter, we'll introduce a number of families of groups.

We'll need a diverse collection of go-to examples to keep us grounded. We'll begin with

1. **cyclic groups**: rotational symmetries
2. **abelian groups**: $ab = ba$
3. **dihedral groups**: rotational *and* reflective symmetries
4. **permutation groups**: collections of rearrangements.

We'll show that *every* finite group is isomorphic to a permutation group.

Then, by modifying some of our familiar groups, we'll encounter the:

5. **quaternion** and **dicyclic groups**,
6. **diquaternion groups**
7. **semidihedral** and **semiabelian groups**.

Finally, we'll take a tour of:

8. **groups of matrices**
9. **direct products** and **semidirect products** of groups.

We'll see a few other visualization techniques and surprises along the way.

A few basic definitions

We'll study subgroups in Chapter 3, but it's helpful to formally define this concept now.

Definition

A subgroup of G is a subset $H \subseteq G$ that is also a group. We denote this by $H \leq G$.

Definition

The order of a group G is its size as a set, denoted by $|G|$.

Definition

The order of an element $g \in G$ is $|g| := \big|\langle g\rangle\big|$, i.e., either

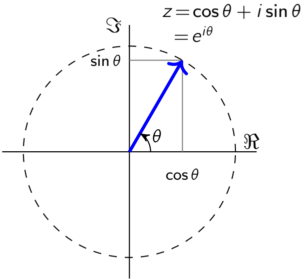
- the minimal $k \geq 1$ such that $g^k = e$, or
- ∞, if there is no such k.

A few basic definitions

The complex numbers are the set

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}, \qquad \text{where } i^2 = -1.$$

By Euler's identity, $e^{i\theta} = \cos\theta + i\sin\theta$ lies on the unit circle.



From this, we get the polar form:

$$z = a + bi = Re^{i\theta}, \qquad \tan\theta = b/a.$$

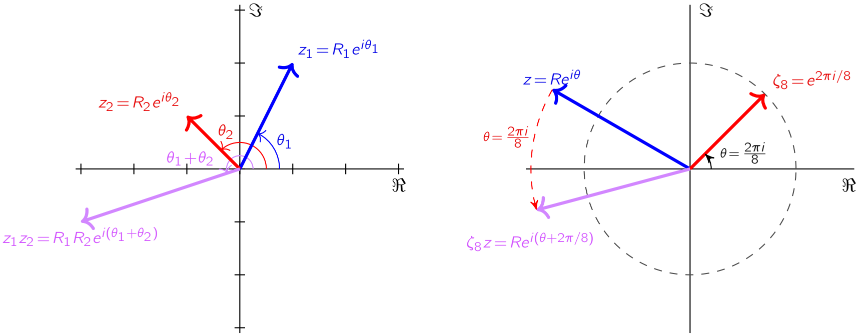
The norm of $z \in \mathbb{C}$ is $|z| := R = \sqrt{a^2 + b^2}$.

Remark

If two complex numbers are multiplied, their lengths multiply and their angles add.

$$z_1 = R_1 e^{\theta_1}, \quad z_2 = R_2 e^{\theta_2} \qquad \Longrightarrow \qquad z_1 z_2 = (R_1 e^{i\theta_1})(R_2 e^{i\theta_2}) = R_1 R_2 e^{i(\theta_1+\theta_2)}.$$

Review of complex numbers



The complex conjugate of $z = Re^{i\theta} = a + bi$ is

$$\overline{z} = Re^{-i\theta} = a - bi,$$

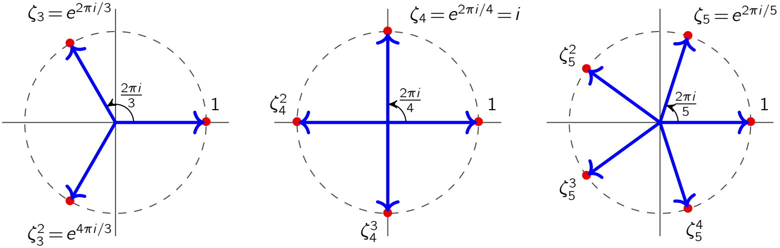
which is the reflection of z across the real axis.

Note that

$$|z|^2 = z \cdot \overline{z} = Re^{i\theta} Re^{-i\theta} = R^2 e^0 = R^2 \quad \Longrightarrow \quad |z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2} = R.$$

Roots of unity

The polynomial $f(x) = x^n - 1$ has n distinct roots, and they lie on the unit circle.



Definition

For $n \geq 1$, the n^{th} roots of unity are the n roots of $f(x) = x^n - 1$, i.e.,

$$U_n := \left\{\zeta_n^k \mid k = 0, \ldots, n-1,\ \zeta_n = e^{2\pi i/n}\right\}.$$

If $\gcd(n, k) = 1$, then ζ_n^k is a primitive n^{th} root of unity.

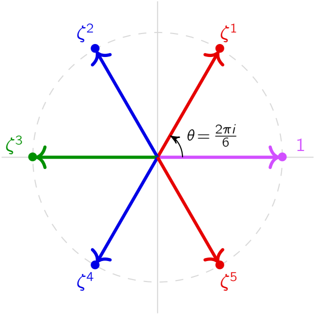
Remark

The n^{th} roots of unity form a group under multiplication.

A motivating example: the 6^{th} roots of unity

The 6^{th} roots of unity are the roots of the polynomial

$$
\begin{aligned}
x^6 - 1 &= (x-1)(x^5+x^4+x^3+x^2+x+1) \\
&= (x-1)(x-e^{2\pi i/6})(x-e^{4\pi i/6})(x-e^{6\pi i/6})(x-e^{8\pi i/6})(x-e^{10\pi i/6}) \\
&= (x-1)(x+1)(x^2+x+1)(x^2-x+1) \\
&= \Phi_1(x)\Phi_2(x)\Phi_3(x)\Phi_6(x)
\end{aligned}
$$



- $\zeta^0 = e^{0\pi i/6} = 1$: primitive 1^{st} root of unity
- $\zeta^1 = e^{2\pi i/6} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$: primitive 6^{th} root of unity
- $\zeta^2 = e^{4\pi i/6} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$: primitive 3^{rd} root of unity
- $\zeta^3 = e^{6\pi i/6} = -1$: primitive 2^{nd} root of unity
- $\zeta^4 = e^{8\pi i/6} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$: primitive 3^{rd} root of unity
- $\zeta^5 = e^{10\pi i/6} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$: primitive 6^{th} root of unity

Do you see how this generalizes for arbitrary n?

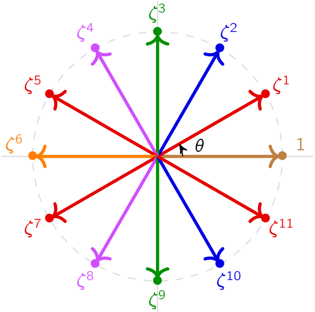
Cyclotomic polynomials

The n^{th} cyclotomic polynomial is $\Phi_n(x) \;:=\; \prod_{\substack{1\leq k<n \\ \gcd(n,k)=1}} (x - e^{2\pi i k/n}) \;=\; \prod_{\substack{1\leq k<n \\ \gcd(n,k)=1}} (x - \zeta_n^k)$.

That is, its roots are precisely the primitive n^{th} roots of unity.

An important fact from number theory is that $\Phi_d(x)$ is irreducible and $x^n - 1 = \prod_{0<d|n} \Phi_d(x)$.

$$\begin{aligned} x^{12}-1 &= \Phi_{12}(x)\,\Phi_6(x)\,\Phi_4(x)\,\Phi_3(x)\,\Phi_2(x)\,\Phi_1(x) \\ &= (x^4-x^2+1)(x^2-x+1)(x^2+1)(x^2+x+1)(x+1)(x-1) \end{aligned}$$



- primitive 12^{th} roots of unity: ζ^1, ζ^5, ζ^7, ζ^{11}
- primitive 6^{th} roots of unity: ζ^2, ζ^{10}
- primitive 4^{th} roots of unity: ζ^3, ζ^9
- primitive 3^{rd} roots of unity: ζ^4, ζ^8
- primitive 2^{nd} root of unity: ζ^6
- primitive 1^{st} root of unity: $\zeta^0 = 1$.

Remark

Primitive d^{th} roots of unity: $\{\zeta^k \mid \gcd(n,k) = n/d\}$.

Reflection matrices

The roots of unity are convenient for representing rotations, but not reflections.

A 2×2 real-valued matrix A is a linear transformation

$$A \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

A reflection across the x-axis (i.e., $v \in V_4$) is the map $(x, y) \mapsto (x, -y)$.

A reflection across the y-axis (i.e., $h \in V_4$) is the map $(x, y) \mapsto (-x, y)$.

In matrix form, these are

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

Multiplying these matrices in either order is $-I$, which is the map $(x, y) \mapsto (-x, -y)$:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Mathematically, this is a representation of the group V_4:

$$V_4 \cong \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Rotation matrices

For $\theta \in [0, 2\pi)$, the rotation matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is a counterclockwise rotation of $\mathbb{R}^2$ about the origin by θ.

Rotating by θ_1 and then by θ_2 is a rotation by $\theta_1 + \theta_2$. Algebraically,

$$A_{\theta_1} A_{\theta_2} = A_{\theta_1+\theta_2}.$$

Recall that multiplication by $e^{2\pi i/n}$ is a counterclockwise rotation of $2\pi/n$ radians in $\mathbb{C}$.

In terms of matrices, this is multiplication by

$$A_{2\pi/n} = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}.$$

We can also represent rotations with complex matrices:

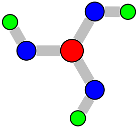
$$R_n := \begin{bmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{bmatrix} = \begin{bmatrix} \zeta_n & 0 \\ 0 & \overline{\zeta}_n \end{bmatrix}.$$

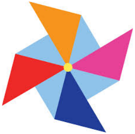
Cyclic groups

Definition

A group is cyclic if it can be generated by a single element.

Finite cyclic groups describe the symmetries of objects that have *only* rotational symmetry.





We have seen three ways to represent cyclic groups.

1. By roots of unity:

$$C_n \cong \langle \zeta_n \rangle = \langle e^{2\pi i/n} \rangle = \left\{ e^{2\pi ik/n} \mid k = 0, \dots n-1 \right\} \subseteq \mathbb{C}.$$

2. By real rotation matrices:

$$C_n \cong \langle A_{2\pi/n} \rangle = \left\langle \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix} \right\rangle .$$

3. By complex rotation matrices:

$$C_n \cong \langle R_n \rangle = \left\langle \begin{bmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & \overline{\zeta}_n \end{bmatrix} \right\rangle .$$

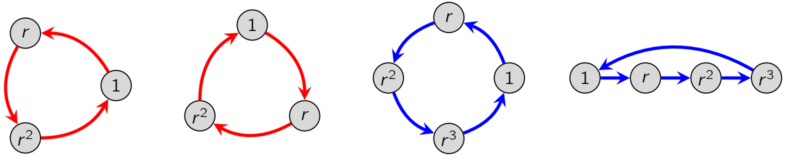
Cyclic groups, multiplicatively

Definition

For $n \geq 1$, the multiplicative cyclic group C_n is the set

$$C_n = \{1, r, r^2, \ldots, r^{n-1}\},$$

where $r^i r^j = r^{i+j}$, and the exponents are taken modulo n. The identity is $r^0 = r^n = 1$.



It is clear that a presentation for this is

$$C_n = \langle r \mid r^n = 1 \rangle.$$

Note that r^2 generates C_5:

$$(r^2)^0 = 1, \quad (r^2)^1 = r^2, \quad (r^2)^2 = r^4, \quad (r^2)^3 = r^6 = r, \quad (r^2)^4 = r^8 = r^3.$$

Do you have a conjecture about for which k does $C_n = \langle r^k \rangle$?

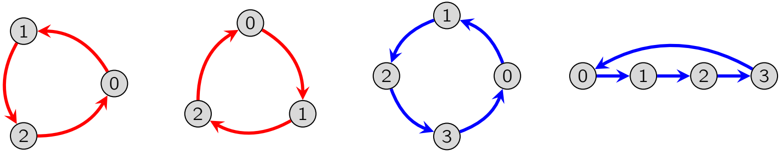
Cyclic groups, additively

Definition

For $n \geq 1$, the additive cyclic group $\mathbb{Z}_n$ is the set

$$\mathbb{Z}_n = \{0, 1, \dots, n-1\},$$

where the binary operation is addition modulo n. The identity is 0.



We can write a group presentation additvely:

$$\mathbb{Z}_n = \langle 1 \mid n \cdot 1 = 0 \rangle.$$

Note that 2 generates $\mathbb{Z}_5$:

$$0 \cdot 2 = 0, \qquad 1 \cdot 2 = 2, \qquad 2 \cdot 2 = 4, \qquad 2 \cdot 3 = 6 \equiv_5 1, \qquad 2 \cdot 4 = 8 \equiv_5 3.$$

Remark

It is wrong to write $C_n = \mathbb{Z}_n$; instead, we say $C_n \cong \mathbb{Z}_n$.

Generators of cyclic groups

Recall that the greatest common divisor of nonzero $a, b \in \mathbb{Z}$ is

$$\gcd(a,b) = \min\left\{|ax+by| : x, y \in \mathbb{Z}\right\},$$

and they are co-prime if $\gcd(a,b)=1$.

Proposition

A number $k \in \{0, 1, \ldots, n-1\}$ generates $\mathbb{Z}_n$ if and only if $\gcd(n,k)=1$.

Equivalently, $C_n = \langle \zeta_n^k \rangle$ if and only if $\zeta_n^k = e^{2\pi i k/n}$ is a primitive n^{th} root of unity.

Proof

"$\Leftarrow$": We need to show that $1 \in \langle k \rangle$.

In other words, that $1 \equiv_n ky$ for some $y \in \mathbb{Z}$.

If $\gcd(n,k)=1$, then write $1 = nx + ky$ for some $x, y \in \mathbb{Z}$. Taking this modulo n yields

$$1 \equiv_n nx + ky \equiv_n ky.$$

We'll leave the "$\Rightarrow$" direction as an exercise. $\square$

Cayley tables of cyclic groups

Modular addition has a nice visual appearance in the Cayley tables for cyclic groups, if we order the elements $0, 1, \ldots, n-1$.

Here are two different ways to write the Cayley table for $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$.

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

	0	1	3	2	4
0	0	1	3	2	4
1	1	2	4	3	0
3	3	4	1	0	2
2	2	3	0	4	1
4	4	0	2	1	3

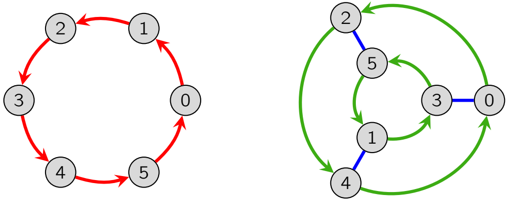
The second Cayley table was one of our mystery Latin square from the previous chapter.

Minimal vs. minimum generating sets

There are many ways to generate the cyclic group of order 6:

$$\mathbb{Z}_6 = \langle 1 \rangle = \langle 5 \rangle = \langle 2,3 \rangle = \langle 3,4 \rangle = \langle 1,2 \rangle = \langle 1,2,3 \rangle = \cdots$$

The following Cayley graphs illustrate two of these.



Definition

Given $G = \langle S \rangle$, the set S is a minimal generating set if $T \subsetneq S$ implies $\langle T \rangle \neq G$.

It is minimum if it is minimal, and if for every other generating set T, we have $|S| \leq |T|$.

Finite groups always have at least one minimum generating set.

What about infinite groups?

Infinite cyclic groups

Definition

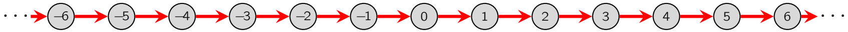
The additive infinite cyclic group is

$$\mathbb{Z} = \langle 1 \mid \quad \rangle,$$

the integers under addition. The multiplicative infinite cyclic group is

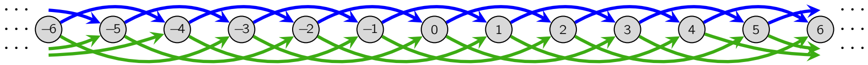
$$C_\infty := \langle r \mid \quad \rangle = \{r^k \mid k \in \mathbb{Z}\}.$$

Several of our frieze groups were cyclic.



There are only two choices for a minimum generating set: $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$.

There are many choices for larger minimal generating sets. Here is $\mathbb{Z} = \langle 2, 3 \rangle$:



Orbits and cycle graphs

Definition

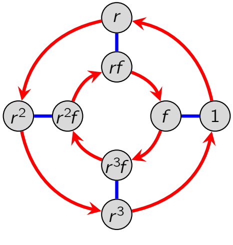
The orbit of an element $g \in G$ is the cyclic subgroup

$$\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\},$$

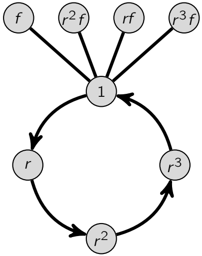
and its order is $|g| := \big|\langle g \rangle\big|$.

We can visualize the orbits by the (undirected) orbit graph, or cycle graph.

This is best seen by an example:



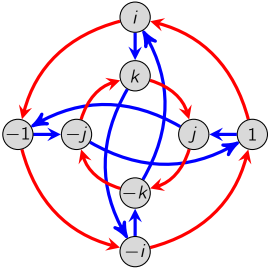
element	orbit
1	$\{1\}$
r^2	$\{1, r^2\}$
r, r^3	$\{1, r, r^2, r^3\}$
f	$\{1, f\}$
rf	$\{1, rf\}$
r^2f	$\{1, r^2f\}$
r^3f	$\{1, r^3f\}$



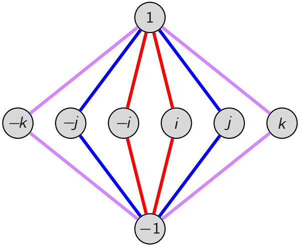
By convention, we typically only draw maximal orbits.

Orbits and cycle graphs

Here is a cycle graph for the quaternion group $Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$.



element	orbit
1	$\{1\}$
-1	$\{\pm 1\}$
i $-i$	$\{\pm 1, \pm i\}$
j $-j$	$\{\pm 1, \pm j\}$
k $-k$	$\{\pm 1, \pm k\}$



Remarks

- We colored the edges to eliminate ambiguity. This is optional, but often helpful.
- We left the edges undirected, because doing so does not introduce ambiguity.
- All of the maximal orbits have size 4.
- All of the size-4 orbits intersect in a size-2 orbit, $\{1, -1\}$.

Dihedral groups

Definition

The dihedral group D_n is the group of symmetries of a regular n-gon. It has order $2n$.

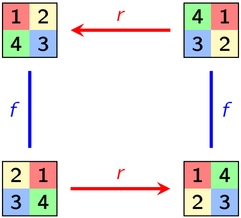
One possible choice of generators is

1. $r =$ counterclockwise rotation by $2\pi/n$ radians,
2. $f =$ flip across a fixed axis of symmetry.

Using these generators, one (of many) ways to write the elements of $D_n = \langle r, f \rangle$ is

$$D_n = \{ \underbrace{1, r, r^2, \dots, r^{n-1}}_{n \text{ rotations}}, \underbrace{f, rf, r^2 f, \dots, r^{n-1} f}_{n \text{ reflections}} \}.$$

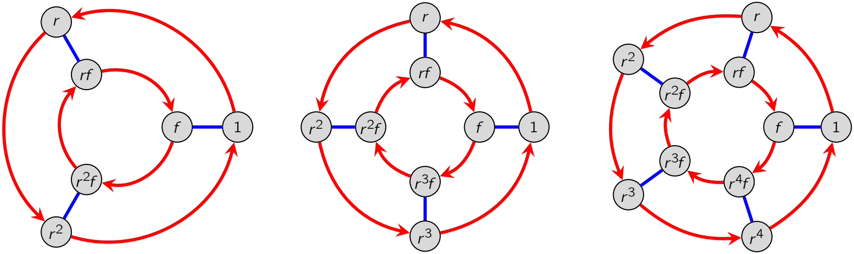
It is easy to check that $rf = fr^{-1}$:



Dihedral groups

Several different presentations for D_n are:

$$D_n = \langle r, f \mid r^n = 1,\ f^2 = 1,\ rfr = f \rangle = \langle r, f \mid r^n = 1,\ f^2 = 1,\ rf = fr^{n-1} \rangle.$$



Warning!

Many books denote the symmetries of the n-gon as D_{2n}.

A strong advantage to our convention is that we can write

$$C_n = \langle r \rangle = \{1, r, r^2, \ldots, r^{n-1}\} \le \langle r, f \rangle = D_n.$$

Dihedral groups

Another canonical way to generate D_n is with two reflections:

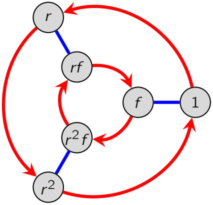
- $s := f$
- $t := fr = r^{n-1}f$

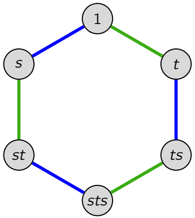
Composing these in either order is a rotation of $2\pi/n$ radians:

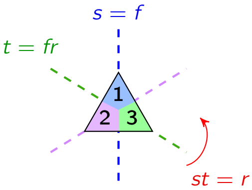
$$st = f(fr) = r, \qquad ts = (fr)f = (r^{n-1}f)f = r^{n-1}.$$

A group presentation with these generators is

$$D_n = \langle s, t \mid s^2 = 1,\ t^2 = 1,\ (st)^n = 1 \rangle = \{ \underbrace{1, st, ts, (st)^2, (ts)^2, \dots,}_{\text{rotations}}\ \underbrace{s, t, sts, tst, \dots}_{\text{reflections}} \}.$$





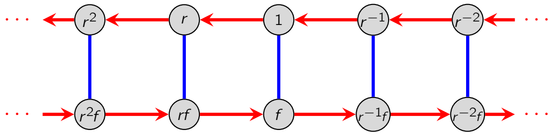


Dihedral groups

Definition

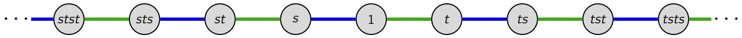
The infinite dihedral group, denoted D_∞, has presentation

$$D_\infty = \langle r, f \mid f^2 = 1,\ rfr = f \rangle.$$



We can also generate D_∞ with two reflections, $s := f$ and $t = fr$.

$$D_\infty = \langle s, t \mid s^2 = 1,\ t^2 = 1 \rangle = \{ \underbrace{1, st, ts, (st)^2, (ts)^2, \dots,}_{\text{“rotations”}} \underbrace{s, t, sts, tst, \dots}_{\text{“reflections”}} \}.$$

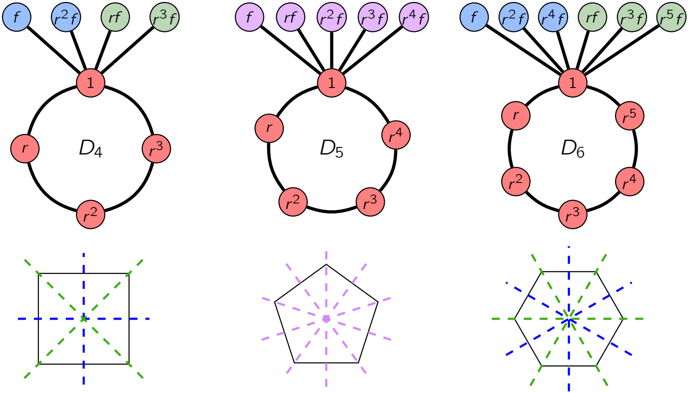


Cycle graphs of dihedral groups

The maximal orbits of D_n consist of

- 1 orbit of size n containing $\{1, r, \dots, r^{n-1}\}$;
- n orbits of size 2 containing $\{1, r^k f\}$ for $k = 0, 1, \dots, n-1$.

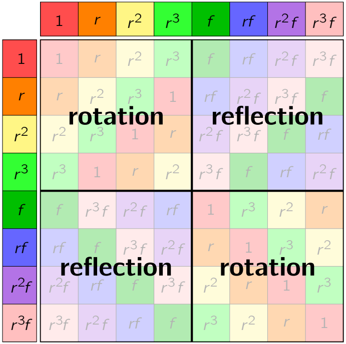
Unless n is prime, the size-n orbit will have smaller subsets that are orbits.



Cayley tables of dihedral groups

The separation of D_n into rotations and reflections is visible in its Cayley tables.

	1	r	r^2	r^3	f	rf	r^2f	r^3f
1	1	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	1	rf	r^2f	r^3f	f
r^2	r^2	r^3	1	r	r^2f	r^3f	f	rf
r^3	r^3	1	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	1	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	1	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	1	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	1



The partition of D_n as depicted above has the structure of group C_2.

"Shrinking" a group in this way is called a quotient.

It yields a group of order 2 with the following Cayley table:

	1	f
1	1	f
f	f	1

Representations of dihedral groups

Recall that the Klein 4-group can be represented by

$$V_4 \cong \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Moreover, a rotation of $2\pi/n$ radians can be

$$A_{2\pi/n} = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix} \qquad \text{or} \qquad R_n := \begin{bmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{bmatrix} = \begin{bmatrix} \zeta_n & 0 \\ 0 & \overline{\zeta}_n \end{bmatrix}.$$

The canonical real representation of D_n with 2×2 matrices is

$$D_n \cong \left\langle \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle.$$

The canonical complex representations of D_n with 2×2 matrices is

$$D_n \cong \left\langle \begin{bmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & \overline{\zeta}_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle.$$

Viewing the groups C_n and D_n as matrices makes our choice of calling the dihedral group D_n (rather than D_{2n}) much more natural!

Abelian groups

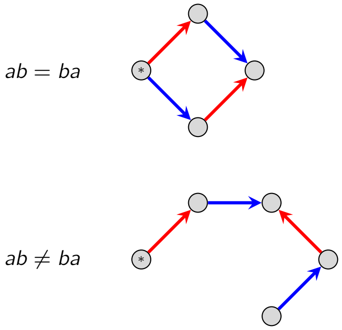
Definition

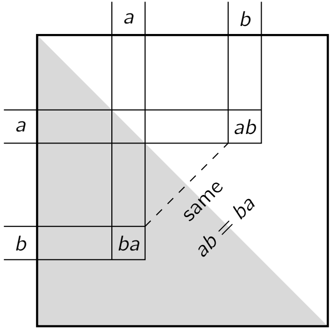
A group G is abelian if $ab = ba$ for all $a, b \in G$.

Remark

To check that G is abelian, it suffices to only check that $ab = ba$ for all pairs of generators.

It is easy to check whether a group is abelian from either its Cayley graph or Cayley table.





Direct products

An easy way to construct finite abelian groups is by taking direct products of cyclic groups.

This is an operation that can be done on any collection of groups.

For two groups, A and B, the Cartesian product is the set

$$A \times B = \big\{(a,b) \mid a \in A,\ b \in B\big\}.$$

Definition

The direct product of groups A and B is the *set* $A \times B$, and the group operation is done component-wise: if $(a,b),(c,d) \in A \times B$, then

$$(a,b) * (c,d) = (ac, bd).$$

We call A and B the factors.

The binary operations on A and B could be different. For example, in $D_4 \times \mathbb{Z}_4$:

$$(rf, 3) * (r^3, 1) = (rfr^3, 1+3) = (r^2 f, 0).$$

These do *not* commute because

$$(r^3, 1) * (rf, 3) = (r^3 rf, 3+1) = (f, 0).$$

Direct products of cyclic groups

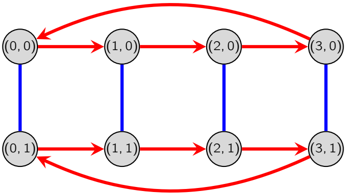
The direct product of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ consists of the set of ordered pairs,

$$\mathbb{Z}_n \times \mathbb{Z}_m = \big\{(a, b) \mid a \in \mathbb{Z}_n,\ b \in \mathbb{Z}_m\big\}.$$

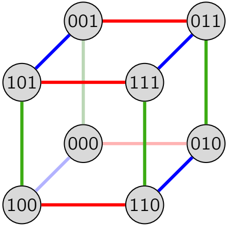
The binary operation is modulo n in the first component, and modulo m in the second component. In other words,

$$(a_1, b_1) + (a_2, b_2) = \big(a_1 + a_2 \pmod{n},\quad b_1 + b_2 \pmod{m}\big).$$

Here are two examples:



$\mathbb{Z}_4 \times \mathbb{Z}_2$

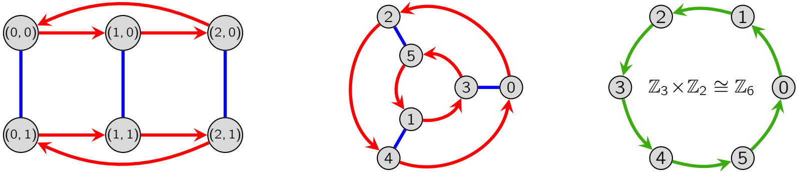


$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong$ **Light**$_3$

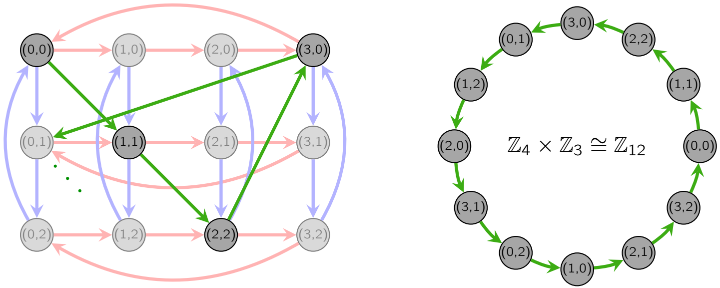
Though $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we will usually write $V_4 \cong C_2 \times C_2$ since we write V_4 multiplicatively.

Direct products of cyclic groups

Sometimes, the direct product of cyclic groups is secretly cyclic.



Here is another example:



Direct products of cyclic groups

Proposition

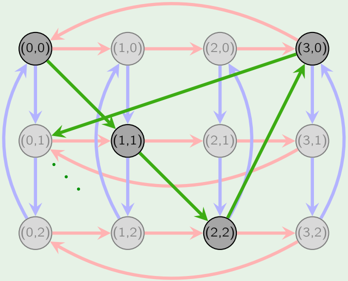
$\mathbb{Z}_{nm} \cong \mathbb{Z}_n \times \mathbb{Z}_m$ if and only if $\gcd(n, m) = 1$.

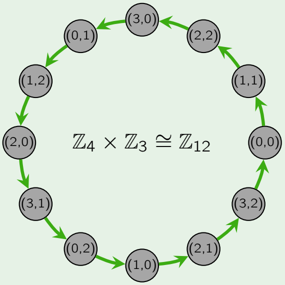
Proof

"$\Leftarrow$": Suppose $\gcd(n, m) = 1$. We claim that $(1, 1) \in \mathbb{Z}_n \times \mathbb{Z}_m$ has order nm.

$|(1, 1)|$ is the smallest k such that "$(k, k) = (0, 0)$." This happens iff $n \mid k$ and $m \mid k$.

Thus, $k = \mathrm{lcm}(n, m) = nm$. ✓





Direct products of cyclic groups

Proposition

$\mathbb{Z}_{nm} \cong \mathbb{Z}_n \times \mathbb{Z}_m$ if an only if $\mathbf{gcd}(n, m) = 1$.

Proof (cont.)

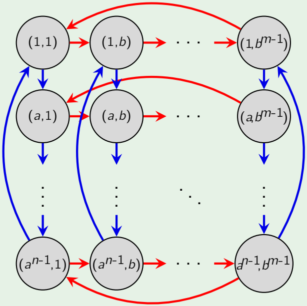
"$\Rightarrow$": Suppose $\mathbb{Z}_{nm} \cong \mathbb{Z}_n \times \mathbb{Z}_m$. Then $\mathbb{Z}_n \times \mathbb{Z}_m$ has an element (a, b) of order nm.

For convenience, we'll switch to "multiplicative notation", and denote our cyclic groups by C_n.

Clearly, $\langle a \rangle = C_n$ and $\langle b \rangle = C_m$. Let's look at a Cayley graph for $C_n \times C_m$.

The order of (a, b) must be a multiple of n (the number of rows), *and* of m (the number of columns).

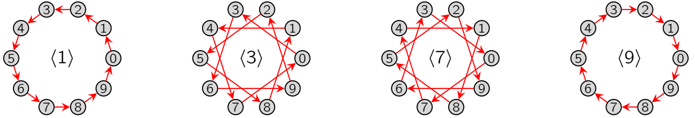
By definition, this is the *least* common multiple of n and m.



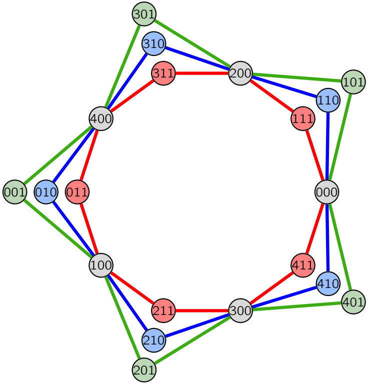
But $|(a, b)| = nm$, and so $\mathbf{lcm}(n, m) = nm$. Therefore, $\mathbf{gcd}(n, m) = 1$. $\square$

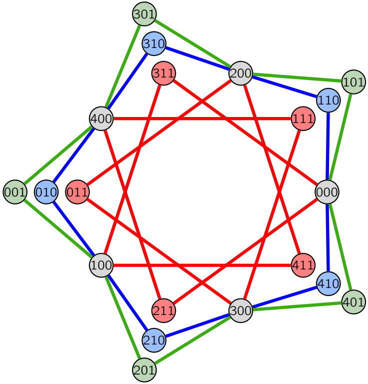
Caveat: cycle graphs need not be unique!

$\mathbb{Z}_{10}$



Both of the following are cycle graphs for $\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.





The fundamental theorem of finite abelian groups

Classification (two different versions)

Every finite abelian group A is isomorphic to a direct product of cyclic groups

$$A \cong \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots \times \mathbb{Z}_{k_m}, \qquad \text{for some } k_1, k_2, \ldots, k_m \in \mathbb{N}, \text{ where}$$

- $k_i = p_i^{d_i}$, for a prime p_i and $d_i \in \mathbb{N}$, ("*prime powers*"), or
- k_i is a multiple of k_{i+1}, ("*elementary divisors*")

Example

Up to isomorphism, there are 6 abelian groups of order $200 = 2^3 \cdot 5^2$:

by "prime-powers"	by "elementary divisors"
$\mathbb{Z}_8 \times \mathbb{Z}_{25}$	$\mathbb{Z}_{200}$
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$	$\mathbb{Z}_{100} \times \mathbb{Z}_2$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$	$\mathbb{Z}_{50} \times \mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_{40} \times \mathbb{Z}_5$
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_{20} \times \mathbb{Z}_{10}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_{10} \times \mathbb{Z}_{10} \times \mathbb{Z}_2$

The fundamental theorem of finitely generated abelian groups

The classification theorem for *finitely generated* abelian groups is not much different.

Theorem

Every finitely generated abelian group A is isomorphic to a direct product of cyclic groups, i.e., for some integers $n_1, n_2, \ldots, n_m$,

$$A \cong \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{k \text{ copies}} \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_m},$$

where each n_i is a prime power, i.e., $n_i = p_i^{d_i}$, where p_i is prime and $d_i \in \mathbb{N}$.

In other words, A is isomorphic to a (multiplicative) group with presentation:

$$A = \langle a_1, \ldots, a_k, r_1, \ldots, r_m \mid r_i^{n_i} = 1,\ a_i a_j = a_j a_i,\ r_i r_j = r_j r_i,\ a_i r_j = r_j a_i \rangle.$$

Non-finitely generated abelian groups that we are familiar with include:

- The *rational numbers*, $\mathbb{Q}$, under addition
- The *real numbers*, $\mathbb{R}$, under addition
- The *complex numbers*, $\mathbb{C}$, under addition
- all of these (with 0 removed) under multiplication: $\mathbb{Q}^*$, $\mathbb{R}^*$, and $\mathbb{C}^*$.
- the positive versions of these under multiplication: $\mathbb{Q}^+$, $\mathbb{R}^+$, and $\mathbb{C}^+$.

Other abelian groups

It is clear that $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$. However, there are many more subgroups of $\mathbb{C}$ than these.

Most of the following are actually rings: additive groups also closed under multiplication. We'll study these more later.

Definition

The Gaussian integers are the complex numbers of the form

$$\mathbb{Z}[i] = \big\{a + bi \mid a, b \in \mathbb{Z}\big\}.$$

We'll see $\mathbb{Z}[\sqrt{-m}]$ and others when we encounter rings of algebraic integers.

The set of polynomials in x "*over the integers*" is a group under addition, denoted

$$\mathbb{Z}[x] = \big\{a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{Z}\big\}.$$

We can also look at certain subgroups, like the polynomials of degree $\leq n$.

Polynomials can be defined in multiple variables, like

$$\mathbb{Z}[x, y] = \Big\{\sum a_{ij} x^i y^j \mid a_{ij} \in \mathbb{Z},\ \text{ all but finitely many } a_{ij} = 0\Big\},$$

or over a finite ring such as $\mathbb{Z}_n$.

Permutation groups

Loosely speaking, a permutation is an action that rearranges a set of objects.

Definition

Let X be a set. A permutation of X is a bijection $\pi \colon X \to X$.

Definition

The permutations of a set X form a group that we denote S_X. The special case when $X = \{1, \ldots, n\}$ is called the symmetric group, and denoted S_n.

If $|X| = |Y|$, then $S_X \cong S_Y$, and so we will usually work with S_n, which has order $n! = n(n-1) \cdots 2 \cdot 1$.

There are several notations for permutations, each with their strengths and weaknesses.

This is best seen with an example:

i	1	2	3	4	5	6
$\pi(i)$	2	3	1	6	5	4

"*one-line notation*"

"*permutation diagram*"

$$\pi = (1\,2\,3)\,(4\,6)$$

"*cycle notation*"

Permutation notations

One-line notation: $\pi = 231654, \qquad \sigma = 564123$

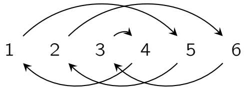
Pros:

- concise
- nice visualization of rearrangement

Cons:

- bad for combining permutations
- not clear where elements get mapped
- hard to compute the inverse

Permutation diagram: π: 

σ: 

Pros:

- can see where elements get mapped
- easy to compute inverses
- convenient for combining permutations

Cons:

- cumbersome to write
- can get tangled

Cycle notation: $\pi = (1\,2\,3)\,(4\,6), \qquad \sigma = (1\,5\,2\,6\,3\,4)$;

Pros:

- short and concise
- easy to see the disjoint cycles
- convenient for combining permutations

Cons:

- representation isn't unique
- not clear what n is

Cycle notation

The cycle $(1\,4\,6\,5)$ means

"*1 goes to 4, which goes to 6, which does to 5, which goes back to 1.*"

Thus, we can write $(1\,4\,6\,5) = (4\,6\,5\,1) = (6\,5\,1\,4) = (5\,1\,4\,6)$.

To find the inverse of a cycle, write it backwards:

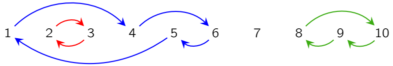
$$(1\,4\,6\,5)^{-1} = (5\,6\,4\,1) = (1\,5\,6\,4) = \cdots$$

Though it's not necessary, we usually prefer to begin a cycle with its smallest number.

Remark

Every permutation in S_n can be written in cycle notation as a product of disjoint cycles, and this is unique up to commuting and cyclically shifting cycles.

For example, consider the following permutation in S_{10}:



as $(1\ 4\ 6\ 5)\,(2\ 3)\,(8\ 10\ 9)$.

This is a product of four disjoint cycles. Since they are disjoint, they commute:

$$(1465)\,(23)\,(8\ 10\ 9) = (23)\,(8\ 10\ 9)\,(1465) = (23)\,(8\ 10\ 9)\,(1465) = \cdots$$

Composing permutations

Remark

The order of a permutation is the least common multiple of the sizes of its disjoint cycles.

For example, $(1\ 3\ 8\ 6)(2\ 9\ 7\ 4\ 10\ 5) \in S_{10}$ has order 12; this should be intuitive.

When cycles are not disjoint, order matters.

Many books compose permutations from right-to-left, due to function composition.

Since we have been using right Cayley graphs, we will compose them from left-to-right.

Notational convention

Composition of permutations will be done left-to-right. That is, given $\pi, \sigma \in S_n$,

$$\pi\sigma \quad \text{means "}\textit{do } \pi, \textit{ then do } \sigma\text{"}.$$

The main drawback about our convention is that it does not work well with function notation applied to elements, like $\pi(i)$.

For example, notice that

$$(\pi\sigma)(i) = \sigma(\pi(i)) \neq \pi(\sigma(i)).$$

However, we will hardly ever use this notation, so that drawback is minimal.

Composing permutations

Here are two ways illustrating how permutations are composed, with the example

First do

i	1	2	3	4	5	6
$\pi(i)$	4	3	1	2	6	5

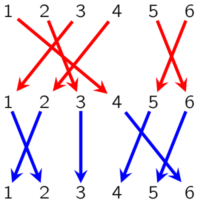
then do

i	1	2	3	4	5	6
$\sigma(i)$	2	1	3	6	4	5

- "By stacking:"

first (1423)(56)

then (12)(465)



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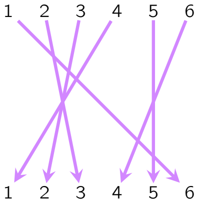
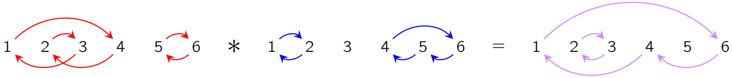


(164)(23)

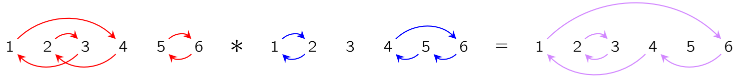
- "By cycles:"



first (1423)(56) *then* (12)(465) (164)(23)

Composing permutations in cycle notation

Let's practice composing two permutations:



first (1423)(56)   *then* (12)(465)   (164)(23)

Let's now do that in slow motion.

In the example above, we start with 1 and then read off:

- "**1** goes to 4, then 4 goes to 6";   Write: (1 6
- "6 goes to 5, then 5 goes to 4";   Write: (1 6 4
- "4 goes to 2, then 2 goes to **1**";   Write: (1 6 4), and start a new cycle.
- "**2** goes to 3, then 3 is fixed";   Write: (1 6 4) (2 3
- "3 goes to 1, then 1 goes to **2**";   Write: (1 6 4) (2 3), and start a new cycle.
- "**5** goes to 6, then 6 goes to **5**";   Write: (1 6 4) (2 3) (5); now we're done.

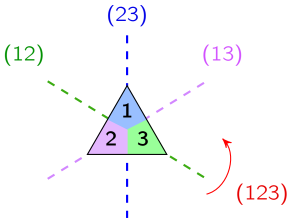
We typically omit 1-cycles (fixed points), so the permutation above is just (1 6 4) (2 3).

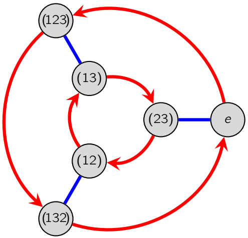
The symmetric group

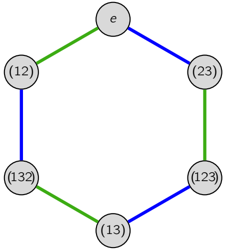
If we number the corners of an n-gon, every symmetry canonically defines a permutation.

However, not every permutation of the corners necessarily is a symmetry, unless $n = 3$.

Indeed, every permutation of $\{1, 2, 3\}$ can be realized as an element of D_3.







Remark

The groups D_n and S_n are isomorphic for $n = 3$, and non-isomorphic if $n > 3$.

The symmetric group

Instead of using configurations of the triangle, consider rearrangements of numbers:

$$\{\mathbf{123},\ \mathbf{132},\ \mathbf{213},\ \mathbf{231},\ \mathbf{312},\ \mathbf{321}\}.$$

Clearly, S_3 canonically rearranges these configurations.

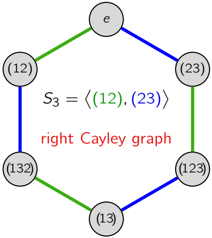
However, *there are two perfectly acceptable interpretations for "canonical."*

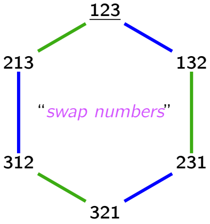
For example, (12) can be interpreted to mean

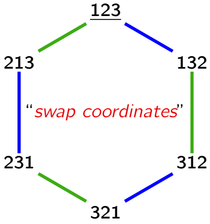
"*swap the numbers in the* $1^{\rm st}$ *and* $2^{\rm nd}$ *coordinates*."

Alternatively, (12) could mean

"*swap the numbers* 1 *and* 2, *regardless of where they are*."







Later, we will understand this difference as a left group action vs. a right group action.

Transpositions

A transposition is a permutation that swaps two objects and fixes the rest, e.g.:

$$\tau = (i\,j): \qquad \begin{tikzcd}[column sep=small] 1 & 2 & \cdots & i-1 & i \arrow[rrrr, bend left] & i+1 & \cdots & j-1 & j \arrow[llll, bend left] & j+1 & \cdots & n-1 & n \end{tikzcd}$$

An adjacent transposition is one of the form $(i \;\; i+1)$.

The following result should be intuitive, if one thinks about rearranging n objects in a row.

Remark

There are two canonical types of generating sets for S_n:

- Adjacent transpositions:

$$S_n = \big\langle (1\,2),\ (2\,3),\ \ldots,\ (n-1 \;\; n) \big\rangle\,.$$

- A transposition and an n-cycle, e.g.,:

$$S_n = \big\langle (1\,2),\ \ (1\;2\;\cdots\;n{-}1\;\;n) \big\rangle\,.$$

Polytopes and platonic solids

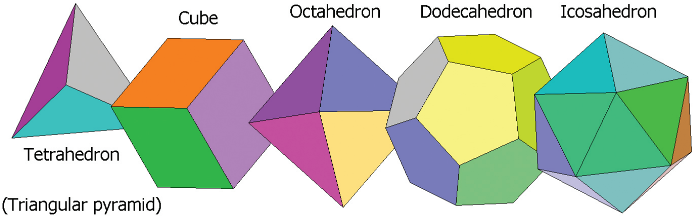
A polytope is a finite region of $\mathbb{R}^n$ enclosed by finitely many hyperplanes.

2D polytopes are *polygons*, and 3D polytopes are polyhedra.

The formal definition of a regular polytope involves a technical condition of its symmetry group.

Informally, it means all faces and all vertices are identical and indistinguishable – higher-dimensional analogues of regular polygons.

There are exactly five regular polyhedra, called Platonic solids.



Archimedean solids

More general than the Platonic solids are the Archimedean solids.

These are non-regular convex uniform polyhedra built from regular polygons.

Though they can involve different polygons, all vertices are locally identical.

In the third century B.C.E., Archimedes classified all 13 such polyhedra.

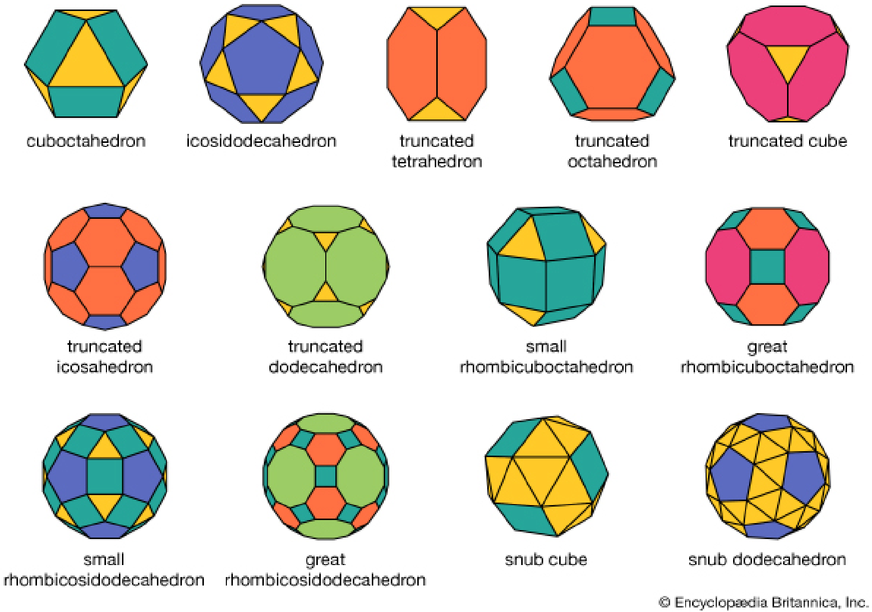
Five are "truncated versions" of the Platonic solids – formed by chopping off vertices.

The others consist of

- the chiral "snub cube" and "snub dodecahedron"
- "hybrids" such as the icosidodecahedron
- truncated versions of these hybrids.

The Cayley graph of S_4 can be arranged on the skeletons of several of these.

Archimedean solids



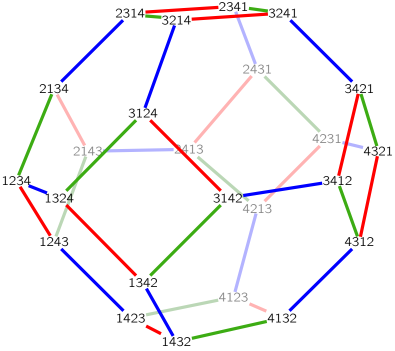
The left and right permutahedra

Definition

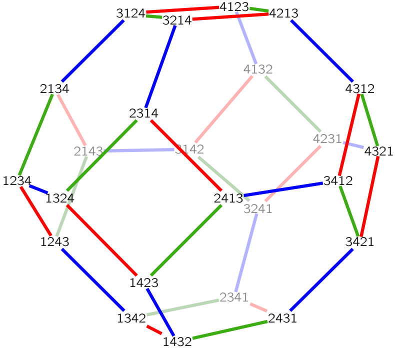
The (right) *n-permutahedron* is the convex hull of the $n!$ permutations of $(1, \dots, n) \in \mathbb{R}^n$.

This is an $(n-1)$-dimensional polytope, as it lies on the hyperplane $x_1 + \cdots + x_n = \frac{(n-1)n}{2}$. It is also the (right) Cayley graph of

$$S_4 = \big\langle (12),\ (23),\ (34) \big\rangle.$$



"*swap coordinates*"



"*swap numbers*"

Even and odd permutations

Remark

Even though every permutation in S_n can be written as a product of transpositions, there may be many ways to do this.

For example: $(132) = (12)(23) = (12)(23)(23)(23) = (12)(23)\,(12)\,(12).$

Proposition

The parity of the number of transpositions of a fixed permutation is unique.

Definition

An even permutation in S_n can be written with an even number of transpositions. An odd permutation requires an odd number.

Remark

The product of:

- two even permutations is even
- two odd permutations is even
- an even an an odd permutation is odd.

The alternating groups

Definition

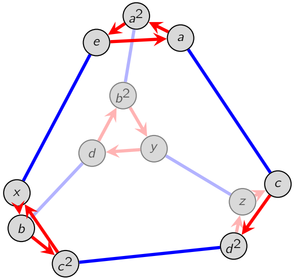
The set of even permutations in S_n is the alternating group, denoted A_n.

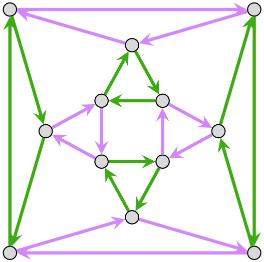
Proposition

Exactly half of the permutations in S_n are even, and so $|A_n| = \frac{n!}{2}$.

Rather than prove this using (messy) elementary methods now, we'll wait until we see the isomorphism theorems to get a 1-line proof.

Here are Cayley graphs for A_4 on a truncated tetrahedron and cuboctahedron.

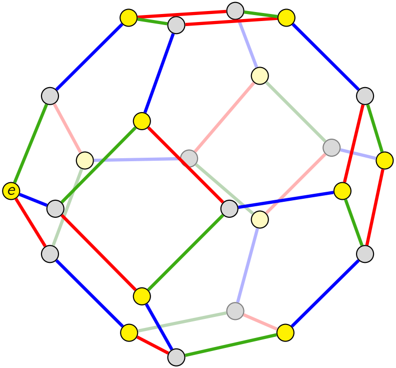




The appearance of A_4 in Cayley graphs for S_4

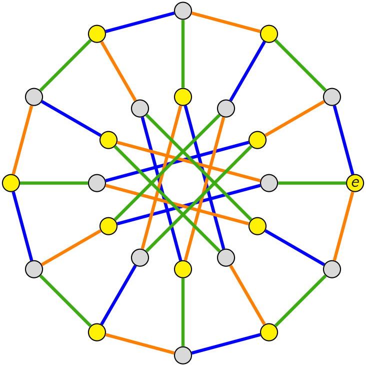
Let's highlight in yellow the even permutations in Cayley graphs for S_4.

$S_4 = \langle (12),\ (23),\ (34) \rangle$



truncated octahedron; "*permutahedron*"

$S_4 = \langle (12), (13), (14) \rangle$



"*Nauru graph*"

Notice that any two paths between yellow nodes has **even length**.

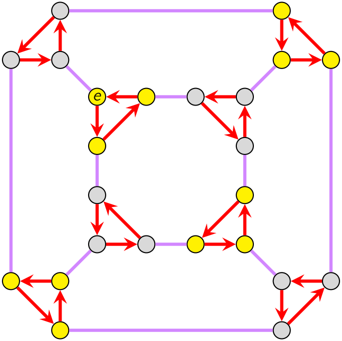
The appearance of A_4 in Cayley graphs for S_4

There are only five **cycle types** in S_4:

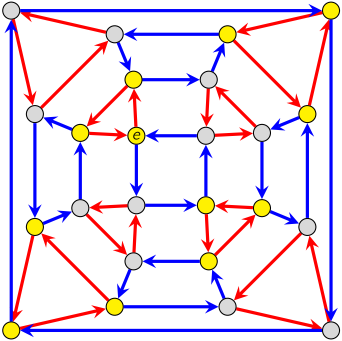
example element	e	(12)	(234)	(1234)	(12)(34)
parity	even	odd	even	odd	even
# elts	1	6	8	6	3

In both Cayley graphs, blue arrows flip the sign of the permutation; red arrows do not.

Once again, even permutations are highlighted in yellow.

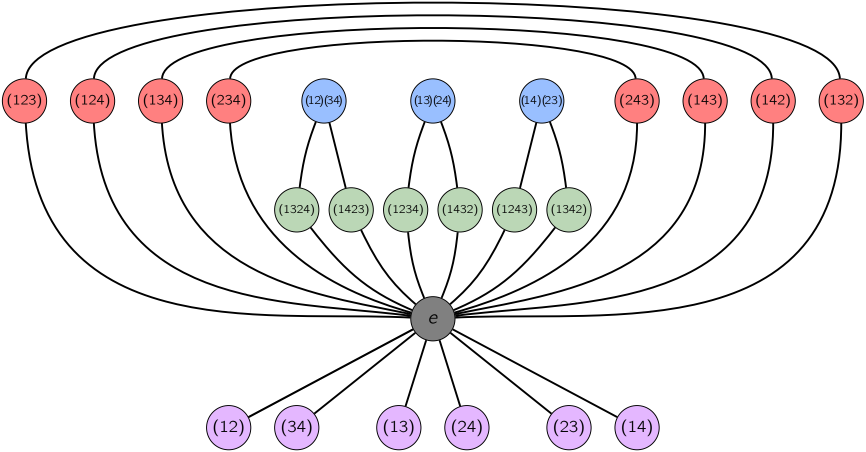


truncated cube



rhombicuboctahedron

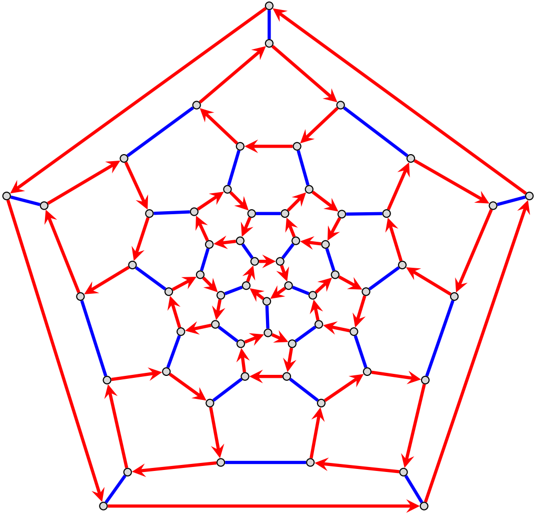
The cycle graph of S_4



A very important group

The group A_5 has special properties that we will learn about later.

Here is the Cayley graph of $A_5 = \langle (12345), (12)(34) \rangle$ on a truncated icosahedron.

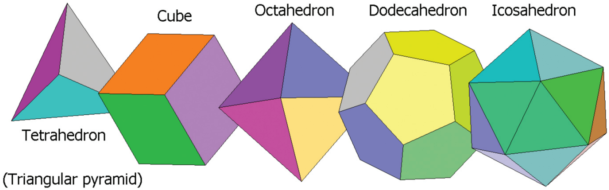


Symmetry groups of Platonic solids

Two-dimensional regular polytopes have rotation groups (C_n) and symmetry groups (D_n).

3D regular polytopes (Platonic solids) have these as well.

solid	**rotation group**	**symmetry group**
Tetrahedron	A_4	S_4
Cube	S_4	$S_4 \times C_2$
Octahedron	S_4	$S_4 \times C_2$
Icosahedron	A_5	$A_5 \times C_2$
Dodecahedron	A_5	$A_5 \times C_2$



There are higher-dimensional versions of the tetrahedron and cube, and their symmetry groups are S_n, and a group we haven't yet seen called $S_n \wr C_2$ (the "signed permutations").

Cayley's theorem

A set of permutations that forms a group is called a permutation group.

A fundamental theorem by British mathematician Arthur Cayley (1821–1895) says that every finite group can be thought of as a collection of permutations.

This is clear for groups of symmetries like V_4, C_n, or D_n, but less so for groups like Q_8.

Cayley's theorem

Every finite group is isomorphic to a collection of permutations, i.e., some subgroup of S_n.

We don't have the mathematical tools to prove this, but we'll get a 1-line proof when we study group actions.

A natural first question to ask is the following:

Given a group, how do we associate it with a set of permutations?

We'll see two algorithms which give strong intuition for why Cayley's theorem is true.

Constructing permutations from a Cayley graph

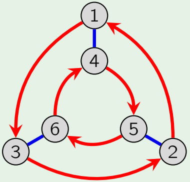
Here is an algorithm given a Cayley graph with n nodes:

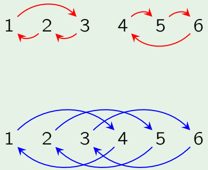
1. number the nodes 1 through n,
2. interpret each arrow type in the Cayley graph as a permutation.

Take the permutations corresponding to the generators.

Example

Let's try this with $D_3 = \langle r, f \rangle$.





We see that D_3 is isomorphic to the subgroup $\big\langle (132)(456),\ (14)(25)(36) \big\rangle$ of S_6.

Constructing permutations from a Cayley table

Here is an algorithm given a Cayley table with n elements:

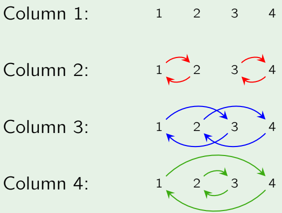
1. replace the table headings with 1 through n,
2. make the appropriate replacements throughout the rest of the table,
3. interpret each column as a permutation.

Take the permutations corresponding to *any* generating set.

Example

Let's try this with the Cayley table for $V_4 = \langle v, h \rangle$.

	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1



We see that V_4 is isomorphic to the subgroup $\big\langle (12)(34),\ (13)(24) \big\rangle$ of S_4.

Permutation matrices

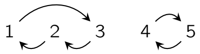
We have seen how to represent groups of symmetries such as V_4, C_n, and D_n as matrices.

Permuting coordinates of $\mathbb{R}^n$ is also a linear transformation.

Every permutation can represented by an $n \times n$ permutation matrix, P_π.

For an example of this, consider the following permutation $\pi \in S_5$:

i	1	2	3	4	5
$\pi(i)$	3	1	2	5	4



$$\pi = (1\,3\,2)\,(4\,5)$$

The matrix P_π permutes the entries of a colum vector:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \\ x_5 \\ x_4 \end{bmatrix},$$

It permutes the entries of a row vector (by coordinates):

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} x_2 & x_3 & x_1 & x_5 & x_4 \end{bmatrix}.$$

Permutation matrices

Definition

Given an element $\pi \in S_n$, the corresponding permutation matrix is the $n \times n$ matrix

$$P_\pi = (p_{ij}), \qquad p_{ij} = \begin{cases} 1 & \pi(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

Here are several more examples of permutation matrices.

$$P_{(12)(34)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad P_{(134)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad P_{(1234)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that the difference between left and right multiplication is:

$P_\pi P_\sigma x$	Right-to-left: "*Start with x, apply σ, then π*"
$x^T P_\pi P_\sigma$	Left-to-right: "*Start with x^T, apply π, then σ*"

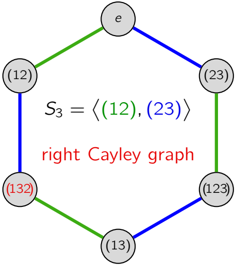
It does not matter whether we use row or column vectors, but we must be careful.

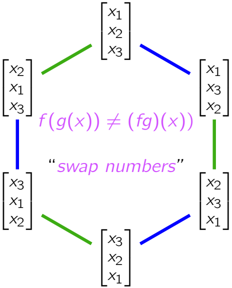
- Column vectors correspond to multiplying right-to-left, as in function composition.
- Row vectors correspond to multiplying left-to-right, which has been our standard.

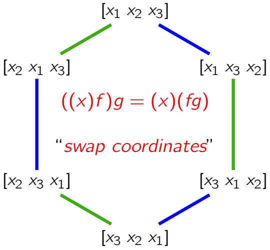
Our left-to-right multiplication convention is more compatible with row vectors

$$P_{(12)}P_{(23)}\mathbf{v} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix} = P_{(132)}\mathbf{v}.$$

$$\begin{aligned} \mathbf{v}^T P_{(12)}P_{(23)} &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} x_2 & x_3 & x_1 \end{bmatrix} = \mathbf{v}^T P_{(132)}. \end{aligned}$$







Generalizing the quaternion group

The **quaternion group** Q_8 is generated by:

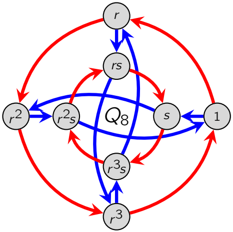
- a 4^{th} root of unity, $i = \zeta_4 = e^{2\pi i/4}$ ($2\pi/4$-rotation)
- the "imaginary number" j

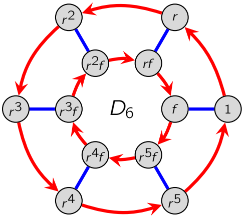
$$Q_8 = \langle i, j, k \rangle \cong \left\langle \underbrace{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{R=R_4}, \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S}, \underbrace{\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}}_{T=RS} \right\rangle.$$

The **dihedral group** is generated by

- an n^{th} root of unity, $r = \zeta_n = e^{2\pi i/n}$ ($2\pi/n$-rotation)
- a reflection f

$$D_n = \langle r, f \rangle \cong \left\langle \underbrace{\begin{bmatrix} \zeta_n & 0 \\ 0 & \bar{\zeta}_n \end{bmatrix}}_{R_n}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{F} \right\rangle.$$



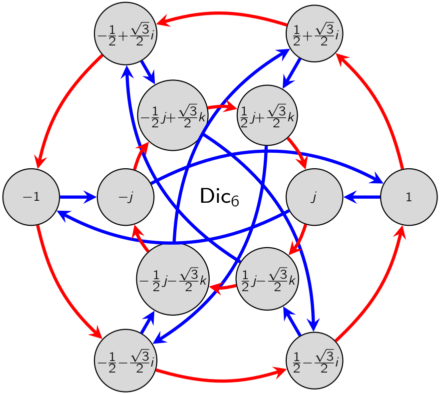


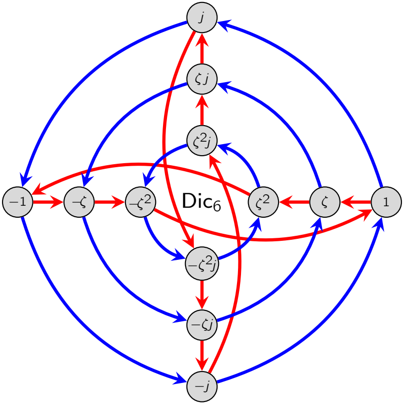
The dicyclic groups

This defines the **dicyclic group**,

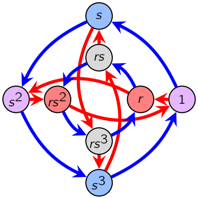
$$\mathrm{Dic}_n = \langle \zeta_n, j \rangle \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & \overline{\zeta_n} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle \cong \langle r, s \mid r^n = s^4 = 1,\ r^{n/2} = s^2,\ rsr = s \rangle.$$

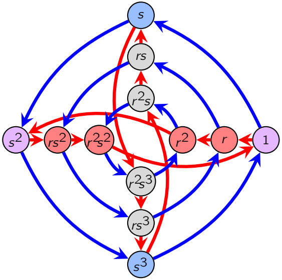
The multiplication rules $ij = k$ and $ji = -k$ remain unchanged.

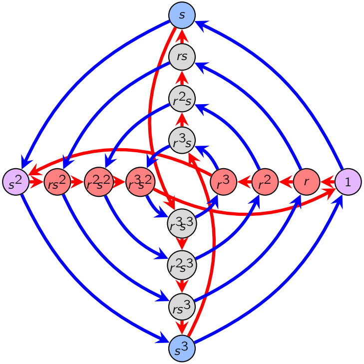


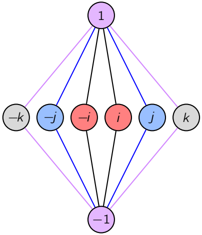


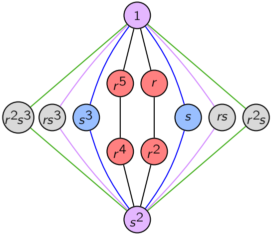
The dicyclic groups

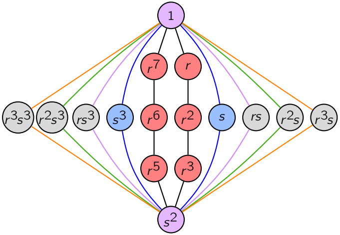








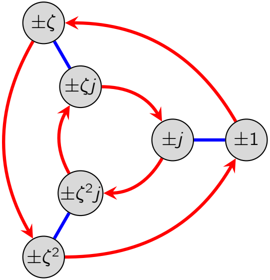




A quotient of the dicyclic group

Recall how we constructed a quotient of the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ that was isomorphic to V_4.

We can do a similar construction for dicyclic groups.



	± 1	$\pm\zeta$	$\pm\zeta^2$	$\pm j$	$\pm\zeta j$	$\pm\zeta^2 j$
± 1	± 1	$\pm\zeta$	$\pm\zeta^2$	$\pm j$	$\pm\zeta j$	$\pm\zeta^2 j$
$\pm\zeta$	$\pm\zeta$	$\pm\zeta^2$	± 1	$\pm\zeta j$	$\pm\zeta^2 j$	$\pm j$
$\pm\zeta^2$	$\pm\zeta^2$	± 1	$\pm\zeta$	$\pm\zeta^2 j$	$\pm j$	$\pm\zeta j$
$\pm j$	$\pm j$	$\pm\zeta^2 j$	$\pm\zeta j$	± 1	$\pm\zeta^2$	$\pm\zeta$
$\pm\zeta j$	$\pm\zeta j$	$\pm j$	$\pm\zeta^2 j$	$\pm\zeta$	± 1	$\pm\zeta^2$
$\pm\zeta^2 j$	$\pm\zeta^2 j$	$\pm\zeta j$	$\pm j$	$\pm\zeta^2$	$\pm\zeta$	± 1

The product $(\pm\zeta j) \cdot (\pm\zeta^2 j) = \pm\zeta^2$ means

"the product of any element in $\{\zeta j, -\zeta j\}$ with any element in $\{\zeta^2 j, -\zeta^2 j\}$ is in $\{\zeta^2, -\zeta^2\}$."

When $n = 2^m$, the dicyclic group $\mathrm{Dic}_{2^{n-1}}$ is called the generalized quaternion group, denoted Q_{2^n}.

The diquaternion group

Recall our standard representations of the quaternion and dihedral groups:

$$Q_8 = \langle i, j, k \rangle \cong \left\langle \underbrace{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{R=R_4}, \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S}, \underbrace{\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}}_{T=RS} \right\rangle, \qquad D_n = \langle r, f \rangle \cong \left\langle \underbrace{\begin{bmatrix} \zeta_n & 0 \\ 0 & \bar{\zeta}_n \end{bmatrix}}_{R_n}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{F} \right\rangle.$$

Now, consider the group generated by adding the reflection matrix from D_n to Q_8.

This is the Pauli group on 1 qubit. We will call it the diquaternion group

$$\mathrm{DQ}_8 = \langle X, Y, Z \rangle = \{ \pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ \},$$

generated by the Pauli matrices from quantum mechanics and information theory:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easy to check that

$$XY = R \quad \text{"}i\text{"}, \qquad XZ = S \quad \text{"}j\text{"}, \qquad YZ = \overline{T} \quad \text{"}-k\text{"}.$$

This group can be constructed in other ways as well:

- as a semidirect product, $Q_8 \rtimes_2 C_2$, and $D_4 \rtimes_2 C_2$, and $(C_4 \times C_2) \rtimes_3 C_2$.
- as the "central product" $\mathrm{DQ}_8 = C_4 \circ D_4$.

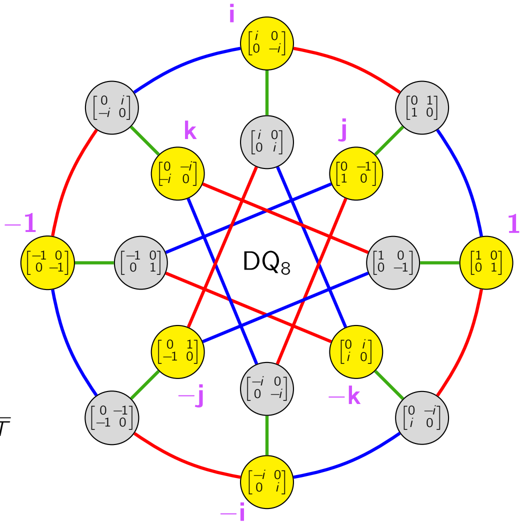
The diquaternion group

$$X = F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

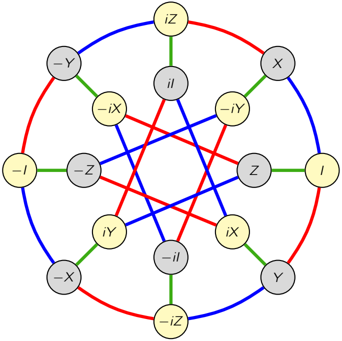
$XY = R, \quad XZ = S, \quad YZ = \overline{T}$

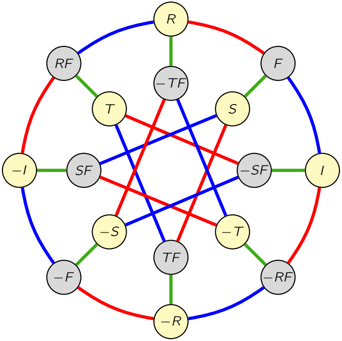


The diquaternion group

The diquaternion group is usually generated with Pauli matrices, $\mathrm{DQ}_8 = \langle X, Y, Z \rangle$.

We can also write it as $\mathrm{DQ}_8 = \langle R, S, T, F\rangle$ where $Q_8 = \langle R, S, T\rangle$ and $D_n = \langle R_n, F\rangle$.



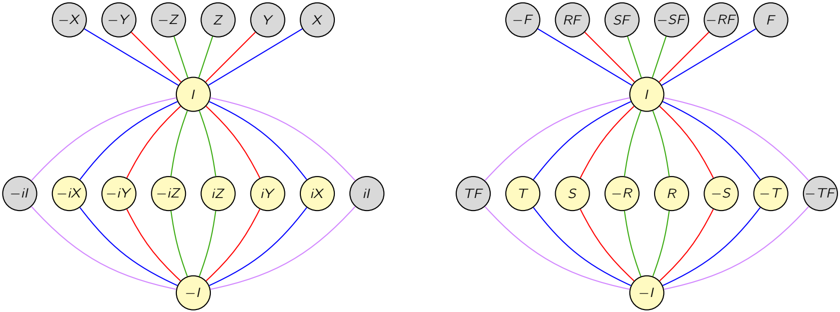


$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad R = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\ S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ T = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

The diquaternion group

Here are two cycle graphs for

$$\mathrm{DQ}_8 = \langle X, Y, Z \rangle = \langle R, S, T, F \rangle.$$



$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \; Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad R = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \; S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; T = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

Do you see a way to generalize this further? What if we use a different root of unity?

Generalized diquaternion groups

Replace $i=\zeta_4=e^{2\pi i/4}$ with $\zeta_n=e^{2\pi i/n}$ to get the generalized diquaternion group

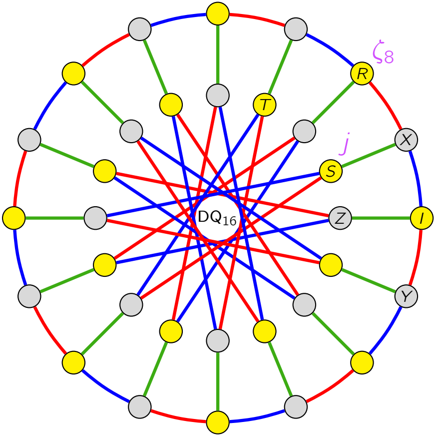
$$\mathrm{DQ}_n := \langle \zeta_n, j, \zeta_n j, f\rangle \cong \left\langle \underbrace{\begin{bmatrix}\zeta_n & 0\\ 0 & \bar{\zeta}_n\end{bmatrix}}_{R=R_n}, \underbrace{\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}}_{S}, \underbrace{\begin{bmatrix}0 & -\zeta_n\\ \bar{\zeta}_n & 0\end{bmatrix}}_{T=T_n}, \underbrace{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}_{F} \right\rangle \cong \mathrm{Dic}_n \rtimes_\theta C_2.$$

$$X = F = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$Y := Y_8 = \begin{bmatrix}0 & \bar{\zeta}_8\\ \zeta_8 & 0\end{bmatrix}$$

$$Z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$XY_8 = R_8, \qquad XZ = S, \qquad Y_8Z = \overline{T}_8$$

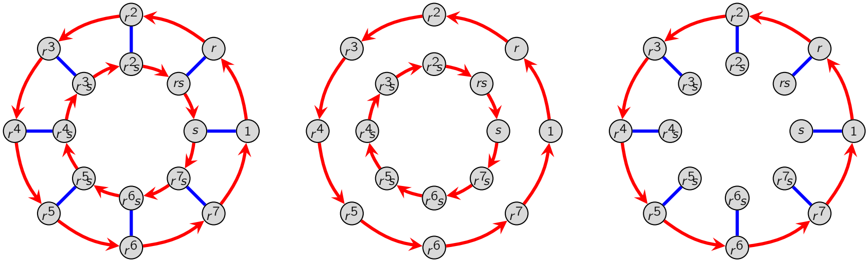


Generalizing the dihedral groups

In our construction of the dicyclic groups, we started with a Cayley graph of $D_n = \langle r, f \rangle$.

We then removed the blue arcs and investigated how we could re-wire them.

But what if we kept those, but re-wired the inner length-n red cycle?



In other words, we want to construct a group G that

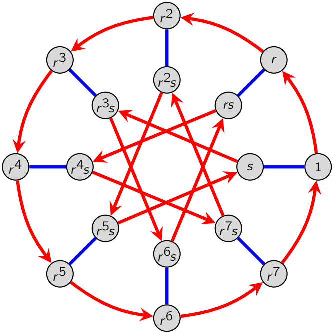
- has an element r of order n
- has an element $s \notin \langle r \rangle$ of order 2.

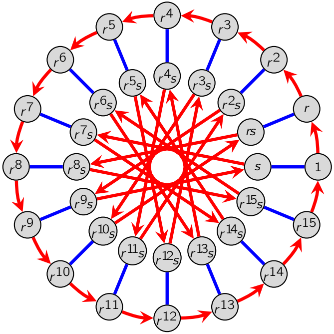
Equivalently, what can we replace the relation $srs = r^{n-1}$ with? That is,

$$G = \left\langle r, s \mid r^n = 1,\, s^2 = 1,\, ??? \right\rangle.$$

Semidihedral groups

If n is a power of 2, we can replace $srs = r^{n-1}$ with $srs = r^{n/2-1}$.





Definition

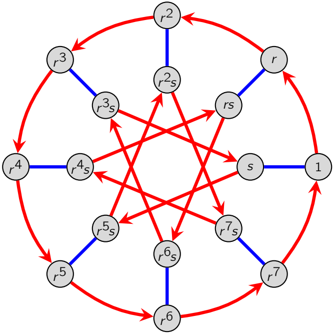
For each power of two, the semidihedral group of order 2^n is defined by

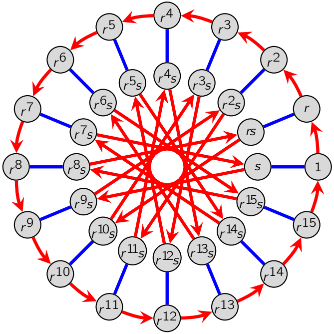
$$\mathrm{SD}_{2^{n-1}} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ srs = r^{2^{n-2}-1} \rangle.$$

Do you see another way we can re-wire these inner red arrows?

Semiabelian groups

Still assuming n is a power of 2, let's replace $srs = r^{n/2-1}$ with $srs = r^{n/2+1}$.





Definition

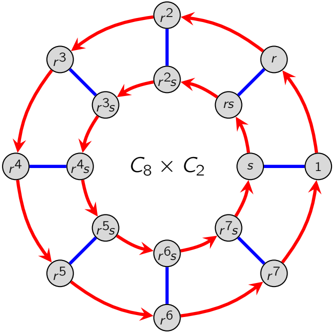
For each power of two, the semiabelian group of order 2^n is defined by

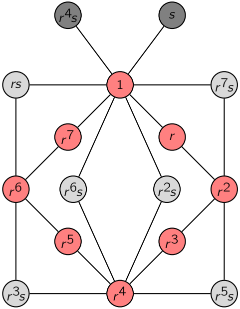
$$\mathrm{SA}_{2^{n-1}} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ srs = r^{2^{n-2}+1} \rangle.$$

One more re-wiring

Of course, there's one more way that we can re-wire the dihedral group. . .

Here is its Cayley graph and cycle graph.





When this group has order 2^n, its presentation is

$$C_{2^{n-1}} \times C_2 = \left\langle r, s \mid r^{2^{n-1}} = s^2 = 1, \, srs = r \right\rangle.$$

Remarkably, this and the other three we've seen are the *only* possibilities:

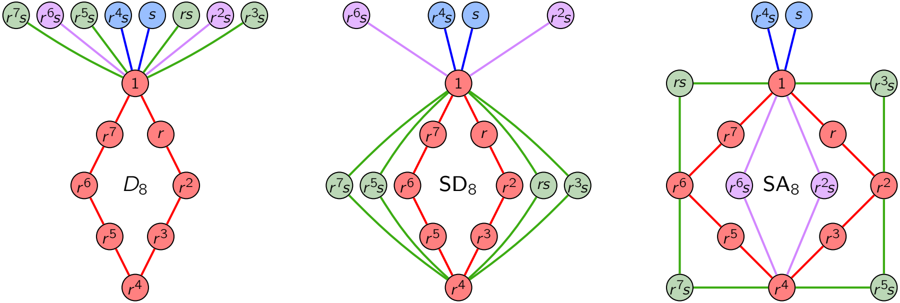
$srs = r^{-1}$ (dihedral), $\quad srs = r^{2^{n-2}-1}$ (semidihedral), $\quad srs = r^{2^{n-2}+1}$ (semiabelian).

Dihedral vs. semidiheral vs. semiabelian groups

In other words, there are exactly 4 groups of order 2^n with both:

- an element r of order 2^{n-1}
- an element $s \notin \langle r \rangle$ of order 2.

Let's compare the cycle graphs of the three non-abelian groups from this list:



Remark

The semiabelian group SA_n and the abelian group $C_n \times C_2$ have the same orbit structure!

This surprising fact has profound consequences that we'll see when we study subgroups.

Dihedral vs. semidiheral vs. semiabelian groups

Compare and contrast representations of the dihedral and semidihedral group:

$$D_n \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & \overline{\zeta}_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle, \qquad \mathrm{SD}_n \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & -\overline{\zeta}_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle, \qquad \zeta_n = e^{2\pi i/n}.$$

Now, compare and contrast those of the abelian and semiabelian group:

$$C_n \times C_2 \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & \zeta_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle, \qquad \mathrm{SA}_n \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & -\zeta_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle.$$

Mnemonic: "semi-" = "*halfway around unit circle*" = $\zeta^{n/2} = -1$.

The groups SD_n and SA_n only exist when $n = 2^m$. In this case, we also have

$$Q_{2^{m+1}} = \mathrm{Dic}_n \cong \left\langle \begin{bmatrix} \zeta_n & 0 \\ 0 & \overline{\zeta}_n \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle,$$

called the generalized quaternion group.

Note that for *any* $n \in \mathbb{N}$, the matrices above generate *some* group.

Exploratory question

What groups do the above representations give if, e.g., n is odd, or not a power of 2?

Non-abelian groups of order 2^n

We'll understand the following better when we study semi-direct products of groups.

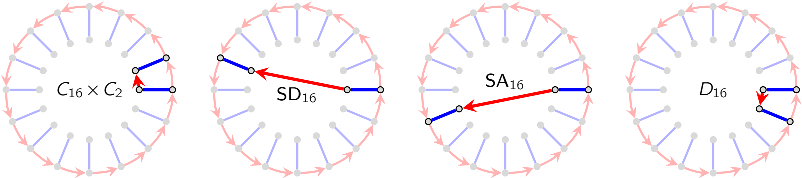
Theorem

There are exactly four nonabelian groups of order 2^n that have an element r of order 2^{n-1}:

1. The **dihedral group** $D_{2^{n-1}} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ srs = r^{-1} \rangle$.
2. The dicyclic group $\mathrm{Dic}_{2^{n-1}} = \langle r, s \mid r^{2^{n-1}} = s^4 = 1,\ r^{2^{n-2}} = s^2,\ rsr = s \rangle$.
3. The semidihedral group $\mathrm{SD}_{2^{n-1}} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ srs = r^{2^{n-2}-1} \rangle$.
4. The semiabelian group $\mathrm{SA}_{2^{n-1}} = \langle r, s \mid r^{2^{n-1}} = s^2 = 1,\ srs = r^{2^{n-2}+1} \rangle$.

As we did before, we can ask:

what groups do these presentations describe when $2n$ is not a power of 2?



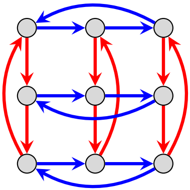
Revisiting direct products

Let A, B be groups with identity elements 1_A and 1_B. Suppose we have a

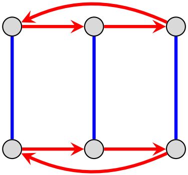
- Cayley graph of A with generators $a_1, \dots, a_k$,
- Cayley graph of B with generators $b_1, \dots, b_\ell$.

We can create a Cayley graph for $A \times B$, by taking

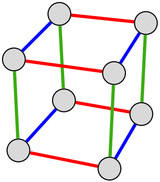
- Vertex set: $\{(a,b) \mid a \in A,\ b \in B\}$,
- Generators: $(a_1, 1_B), \dots, (a_k, 1_B)$ and $(1_A, b_1), \dots, (1_A, b_\ell)$.



$C_3 \times C_3$



$C_3 \times C_2$



$C_2 \times C_2 \times C_2$

Remark

"A-arrows" are independent of "B-arrows." Algebraically, this means

$$(a, 1_B) * (1_A, b) = (a, b) = (1_A, b) * (a, 1_B).$$

Revisiting direct products

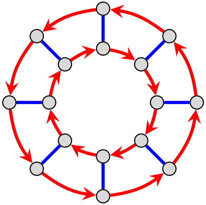
Remark

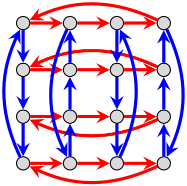
Just because a group is not written with $\times$ does not mean that there is not secretly a direct product structure lurking behind the scenes.

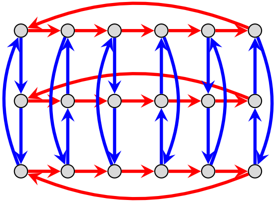
We have already seen that $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and that $C_6 \cong C_3 \times C_2$.

However, sometimes it is even less obvious.

Two of the following three groups secretly have a direct product structure.







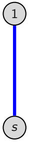
(And it's probably not the two you think.)

Semidirect products

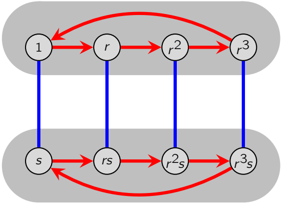
Semidirect products are a more general construction than the direct product.

They can be thought of as a "twisted" version of the direct product.

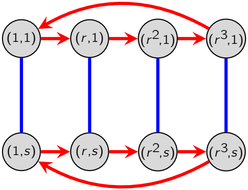
To motivate this, consider the following "inflation method" for constructing the Cayley graph of a direct product:



Start with a copy of $B = C_2$



Inflate each node, insert $A = C_4$ in each and connect corresponding nodes with edges



"pop" each inflated node to get the direct product $C_4 \times C_2$

Consider this process, but with the red arrows reversed in the bottom inflated node.

This would result in a Cayley graph for the group D_4.

We say that D_4 is the semidirect product of C_4 and C_2, written $D_4 \cong C_4 \rtimes C_2$.

Semidirect products

Reversing the red arrows worked is because it was a structure-preserving rewiring.

Formally, this is an automorphism, which is an isomorphism from a group to itself.

We'll learn more about this when we study homomorphisms. Just know that it's a bijection

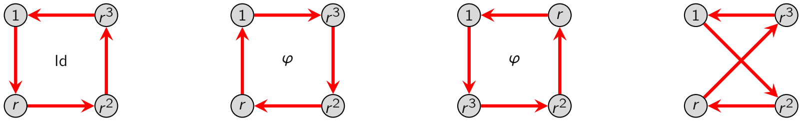
$$\varphi \colon G \longrightarrow G$$

satisfying some extra properties.

There are two ways to describe a rewiring:

- fix the position of the nodes and rewire the edges
- fix the position of the edge and relabel the nodes.

This is best seen with an example:



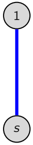
Cayley graph of C_4 | edges rewired | nodes relabeled | non-rewiring

The graph on the right isn't allowed because it doesn't preserve the algebraic structure.

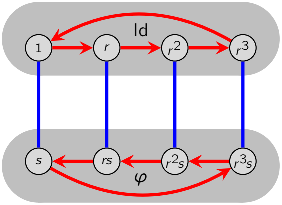
Semidirect products

Semidirect products can be constructed via the "inflation process" for $A \times B$, but *insert φ-rewired copies of the Cayley graph for A into inflated nodes of B*.

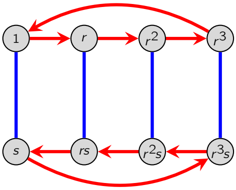
Let's construct $A \rtimes B$ for $A = C_4$ and $B = C_2$, with the rewiring φ from the previous slide.



Start with a copy of $B = C_2$



Inflate each node, insert rewired versions of $A = C_4$, and connect corresponding nodes



"pop" each inflated node to get the semidirect product $C_4 \rtimes_\varphi C_2 \cong D_4$

In the middle graph, each inflated node of $B = C_2 = \langle s \rangle$ is labeled with a re-wiring.

Formally, this is a just map

$$\theta \colon C_2 \longrightarrow \mathrm{Aut}(C_4), \qquad \theta(g) = \begin{cases} \mathrm{Id} & g = 1 \\ \varphi & g = s, \end{cases}$$

where $\theta(g)$ specifies which re-wiring gets put into the inflated node g of C_2.

Semidirect products

There are strong restrictions for inserting rewirings of the Cayley graph of A into B.

The map θ must be a structure-preserving map, called a homomorphism.

If we stick a φ-rewiring into the inflated node $b \in B$, then we must insert a φ^2-rewiring into node $b^2 \in B$, and so on.

Definition (informal)

Consider groups A, B, and a structure-preserving map

$$\theta \colon B \longrightarrow \mathrm{Aut}(A)$$

to the set of rewirings of A. The **semidirect product** $A \rtimes_\theta B$, is constructed by:

- inflating the nodes of the Cayley graph of B, [*mnemonic*: B for "ballooon"]
- inserting a $\theta(b)$-rewiring of the Cayley graph A into node b of B,
- For each edge bewteen B-nodes, connect corresponding pairs of A-nodes with that edge.

Semidirect products

Key point

For groups A, B and map

$$\theta\colon B \longrightarrow \mathrm{Aut}(A),$$

the image $\theta(b)$ can be thought of as "*which rewiring node $b \in B$ gets label with*".

Any group A always has a trivial rewiring.

Remark

For the trivial map $\theta\colon B \longrightarrow \mathrm{Aut}(A)$ sending everything to the identity rewiring

$$A \rtimes_\theta B = A \times B.$$

For any n, there is a rewiring φ of $C_n = \langle r \rangle$ that "reverses all of the r-arrows".

The semidirect product of C_n and $C_2 = \{1, s\}$, with respect to

$$\theta\colon C_2 \longrightarrow \mathrm{Aut}(C_n), \qquad \theta(g) = \begin{cases} \mathrm{Id} & g = 1 \\ \varphi & g = s, \end{cases}$$

is $D_n \cong C_n \rtimes_\theta C_2$.

Semidirect products

Reasons for introducing semidirect products this early

- it helps us understand a new way to construct groups
- it helps us understand the structure of some groups we've already seen
- thinking about *what* works in this process and *why*, helps us gain a more holistic understanding about group theory
- it will be easier to learn advanced concepts such as automorphisms if we get a preview of them in advance, and gain intutition

Proposition

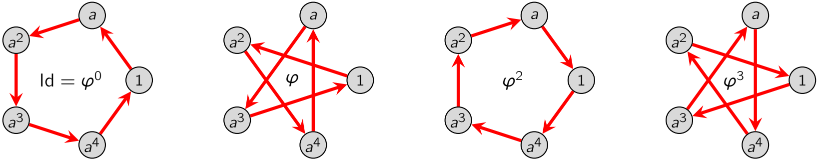
The set of rewirings of a Cayley graph of G forms a group, denoted $\mathsf{Aut}(G)$.

Moreover, this group does not depend on the Cayley graph, but on the group itself.

Rewirings and the automorphism group

There are four rewirings (i.e., automorphisms) of the Cayley graph of $C_5 = \langle a \rangle$.

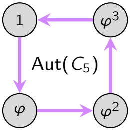
Every rewiring can be realized by iterating the "doubling map" $\varphi \colon C_5 \to C_5$ that replaces each instance of a with a^2, i.e., a length-k path with a length-$2k$ path.

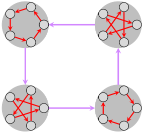


starting graph $\qquad a^1 \mapsto (a^1)^2 = a^2 \qquad a^2 \mapsto (a^2)^2 = a^4 \qquad a^4 \mapsto (a^4)^2 = a^3$

Notice that the rewirings form a group:

$$\mathrm{Aut}(C_5) = \{1,\, \varphi,\, \varphi^2,\, \varphi^3\} \cong C_4$$





Remark

For any group G, the set $\mathrm{Aut}(G)$ of rewirings forms a group, called its automorphism group.

The automorphism group of C_n

Each automorphism is defined by where it sends a generator: $r \mapsto r^k$.

"*each red arrow gets multiplied by k*"

The group $\mathsf{Aut}(C_n)$ is isomorphic to the group with operation multiplication modulo n:

$$U_n := \{k \mid 0 < k < n,\ \gcd(n,k) = 1\}.$$

Example:

$$\mathsf{Aut}(C_7) \cong U_7 = \{1,2,3,4,5,6\} = \langle 3 \rangle \cong C_6$$

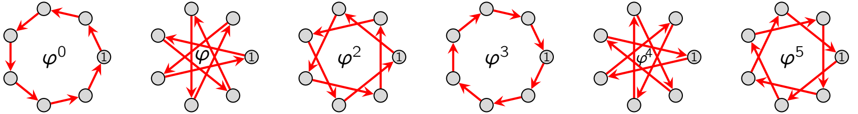
$$2^0 = 1, \quad 2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 1$$

$$3^0 = 1, \quad 3^1 = 3, \quad 3^2 = 2$$

$$3^3 = 6, \quad 3^4 = 4, \quad 3^5 = 5$$

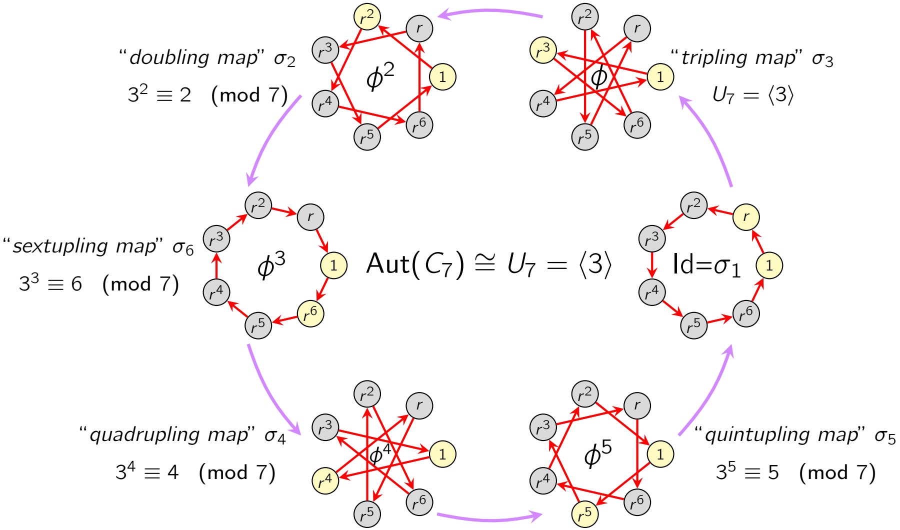
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Since $U_7 = \langle 3 \rangle$, the re-wirings of C_7 are generated by the "*tripling map*" $r \xmapsto{\varphi} r^3$.



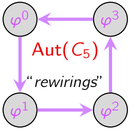
$C_7 = \langle r \rangle$ $\quad r^1 \mapsto (r^1)^3 = r^3$ $\quad r^3 \mapsto (r^3)^3 = r^2$ $\quad r^2 \mapsto (r^2)^3 = r^6$ $\quad r^6 \mapsto (r^6)^3 = r^4$ $\quad r^4 \mapsto (r^4)^3 = r^5$

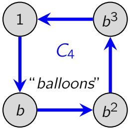
An example: the automorphism group of C_7



An example: the 1^{st} semidirect product of C_5 and C_4

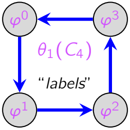
Let's construct a semidirect product $C_5 \rtimes_{\theta_1} C_4$:



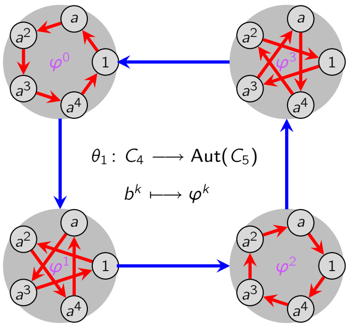


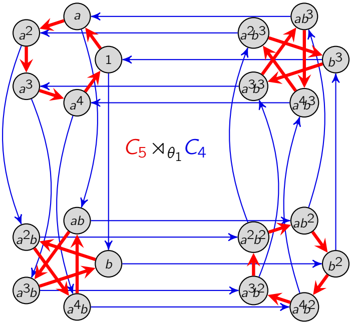
"*labeling map*"

$$C_4 \xrightarrow{\theta_1} \text{Aut}(C_5)$$
$$b^k \longmapsto \varphi^k$$



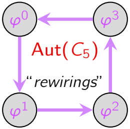
Stick in rewired copies of A, and then reconnect the B-arrows.

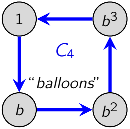




An example: the 2^{nd} semidirect product of C_5 and C_4

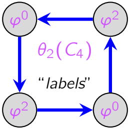
Let's now construct a different semidirect product, $C_5 \rtimes_{\theta_2} C_4$:



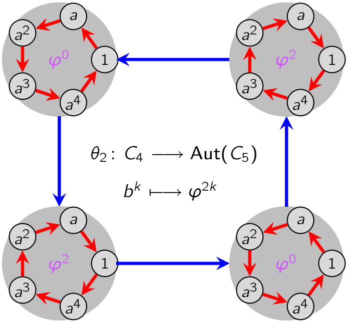


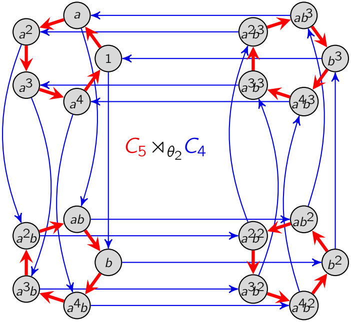
"*labeling map*"

$$C_4 \xrightarrow{\theta_2} \mathrm{Aut}(C_5)$$
$$b^k \longmapsto \varphi^{2k}$$



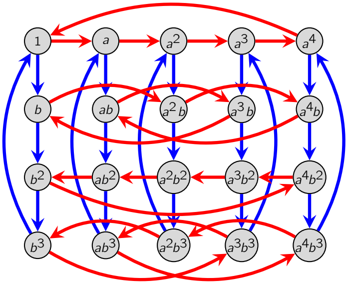
Stick in rewired copies of A, and then reconnect the B-arrows.

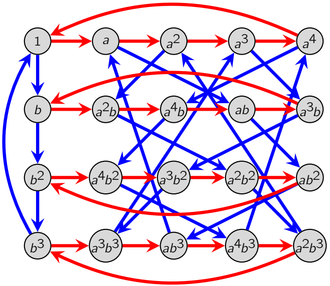




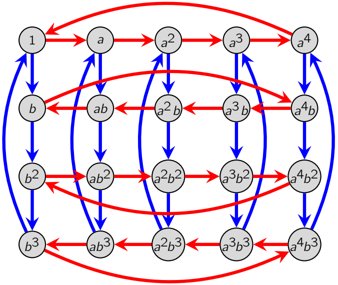
Rewiring edges vs. re-labeling nodes

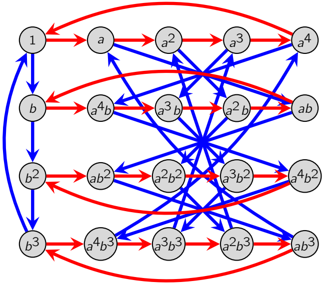
$C_5 \rtimes_{\theta_1} C_4$





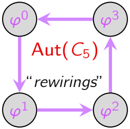
$C_5 \rtimes_{\theta_2} C_4$

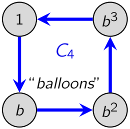




An example: the 3rd semidirect product of C_5 and C_4

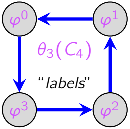
Let's construct another semidirect product $C_5 \rtimes_{\theta_3} C_4$:



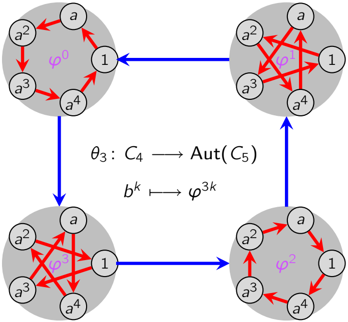


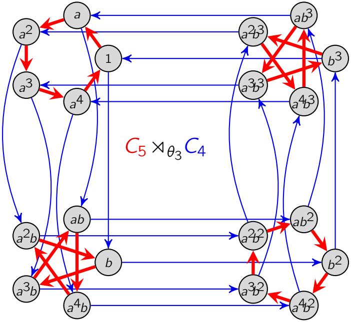
"*labeling map*"

$$C_4 \xrightarrow{\theta_3} \mathrm{Aut}(C_5)$$
$$b^k \longmapsto \varphi^{3k}$$



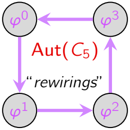
Stick in rewired copies of A, and then reconnect the B-arrows.

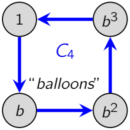




An example: the direct product of C_5 and C_4

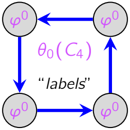
Let's now construct the "trivial" semidirect product, $C_5 \rtimes_{\theta_0} C_4 = C_5 \times C_4$:



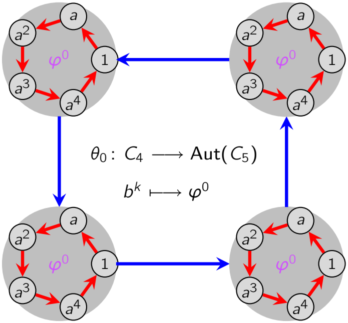


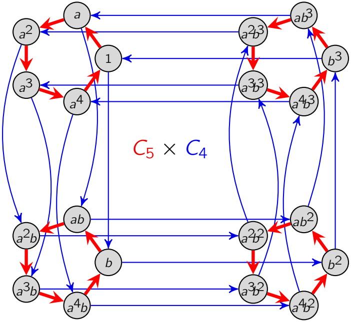
"*labeling map*"

$$C_4 \xrightarrow{\theta_0} \mathrm{Aut}(C_5)$$
$$b^k \longmapsto \varphi^0$$



Stick in rewired copies of A, and then reconnect the B-arrows.

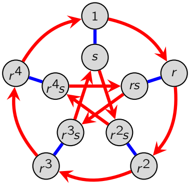




Semidirect products

Questions

- does our semidirect product construction actually yield a group?
- (what would happen if we try C_5 and C_2?)
- when do 2 labeling maps give isomorphic semidirect products?
- is the semidirect product commutative?



not a group

Which groups did we encounter when constructing $C_5 \rtimes_{\theta_k} C_4$, for $k = 1, 2, 3$?

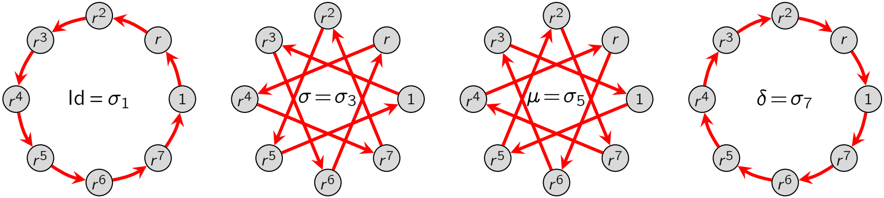
It turns out that there are only three nonabelian groups of order 20:

1. the dihedral group D_{10}
2. the dicyclic group Dic_{10}
3. a 1D "*affine group*" $\mathrm{AGL}_1(\mathbb{Z}_5) \cong \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{Z}_5,\ a \neq 0 \right\} \leq \mathrm{GL}_2(\mathbb{Z}_5)$.

We'll answer these questions and more later, when we study automorphisms.

Semidirect products of C_8 and C_2

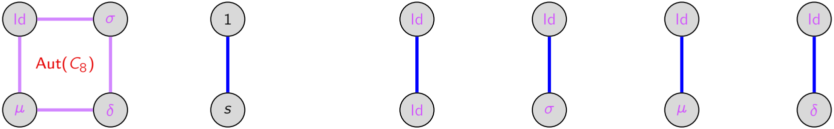
There are four rewirings of the Cayley graph $C_8 = \langle r \rangle$:



All three non-trivial rewirings have order 2:

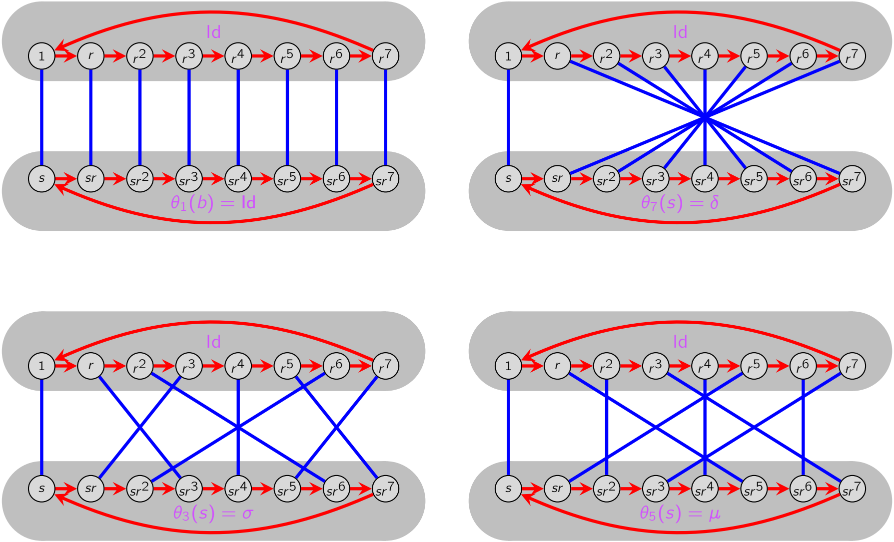
$$r \xrightarrow{\sigma} r^3 \xrightarrow{\sigma} (r^3)^3 = r^9 = r, \qquad r \xrightarrow{\mu} r^5 \xrightarrow{\mu} (r^5)^5 = r^{25} = r, \qquad r \xrightarrow{\delta} r^7 \xrightarrow{\delta} (r^7)^7 = r^{49} = r.$$

There are four labeling maps $\theta_k \colon C_2 \longrightarrow \mathrm{Aut}(C_8) \cong V_4$:



$s \overset{\theta_1}{\longmapsto} \mathrm{Id}$ $\qquad$ $s \overset{\theta_3}{\longmapsto} \sigma$ $\qquad$ $s \overset{\theta_5}{\longmapsto} \mu$ $\qquad$ $s \overset{\theta_7}{\longmapsto} \delta$

The four semidirect products $C_8 \rtimes_i C_2$



Semidirect products of C_{2^m} and C_2

Theorem

For each $n = 2^m$, there are four distinct semidirect products of C_n with C_2:

1. $C_n \rtimes_{\theta_1} C_2 \cong C_n \times C_2$,
2. $C_n \rtimes_{\theta_\sigma} C_2 \cong \mathrm{SD}_n$,
3. $C_n \rtimes_{\theta_\mu} C_2 \cong \mathrm{SA}_n$,
4. $C_n \rtimes_{\theta_\delta} C_2 \cong D_n$,

where the rewirings are maps $C_{2^m} \to C_{2^m}$ defined by

$$r \overset{\theta_1}{\longmapsto} r, \qquad r \overset{\theta_\sigma}{\longmapsto} r^{2^{m-1}-1}, \qquad r \overset{\theta_\mu}{\longmapsto} r^{2^{m-1}+1}, \qquad r \overset{\theta_\delta}{\longmapsto} r^{-1}.$$

The reason why this holds is that $\theta(b)$ in $\mathrm{Aut}(C_{2^m})$ must be an order of order 1 or 2, because $\theta(b^2) = \theta(1) = \mathrm{Id}$.

There are only three elements of order 2 in the group $U(C_{2^m})$, due to the following result from number theory.

Lemma

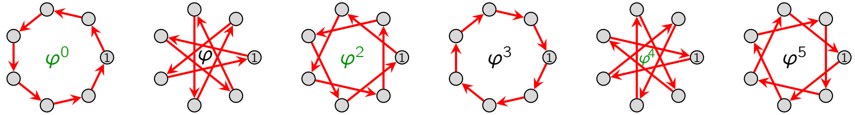
For any $n \geq 3$, the quadratic equation

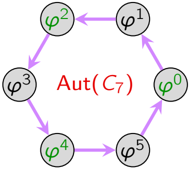
$$x^2 \equiv 1 \pmod{2^n}$$

has exactly four distinct solutions, ± 1 and $2^{n-1} \pm 1$.

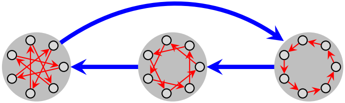
The smallest nonabelian group of odd order: $C_7 \rtimes_\theta C_3$

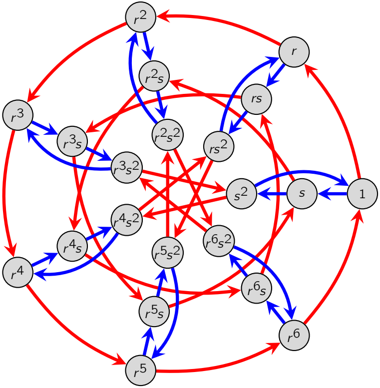
There are 6 re-wirings (automorphisms) of C_7:





$$C_3 \xrightarrow{\theta} \mathrm{Aut}(C_7)$$
$$s^k \longmapsto \varphi^{2k}$$



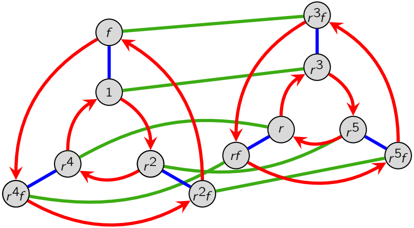


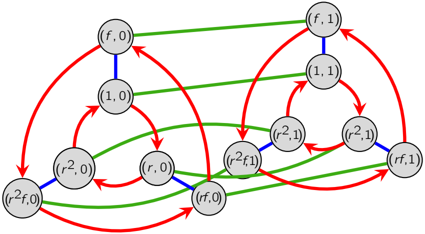
A surprising fact

We know that we can construct the dihedral group D_6 as a semidirect product $C_6 \rtimes_\theta C_2$.

But it also secretly decomposes as a *direct product*!

To see this, let's draw a Cayley graph with a nonstandard generating set, $D_6 = \langle r^2, r^3, f \rangle$.





It is apparent that $D_6 \cong D_3 \times \mathbb{Z}_2 = \langle (r, 0), (f, 0), (0, 1) \rangle$!

Question: *How does this generalize to larger dihedral groups?*

We'll understand this better later when we study subgroups.

Groups of matrices

We have already seen how many familiar groups can be represented by matrices.

Matrices are a rich source of groups in their own right.

Let's define a few terms so we can better speak of certain sets of matrices.

Square matrices are objects that we can add, subtract, and multiply, but not always divide.

Definition

A ring is an abelian group R that is additionally

- closed under multiplication, and
- satisfies the distributive property.

If we can also divide by any nonzero element, it is a field, $\mathbb{F}$.

Some rings contain zero divisors: two nonzero x, y such that $xy = 0$.

For example, $2 \cdot 3 = 0$ in $\mathbb{Z}_6$.

In other rings, multiplication does not commute.

Henceforth, we will assume that our matrix coefficients m_{ij} come from a field $\mathbb{F}$.

Basically, we're intersted in examples like $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$, etc.

Groups of matrices

The set $\mathsf{Mat}_{n,m}(\mathbb{F})$ of $n \times m$ matrices is a group under addition, but a very boring one.

It is isomorphic to the direct product $\mathbb{F}^{mn} := \mathbb{F} \times \cdots \times \mathbb{F}$ of nm copies of $\mathbb{F}$.

It is more interesting to look at groups of square matrices under multiplication.

Definition

Let $\mathsf{Mat}_n(\mathbb{F})$ be the set of $n \times n$ matrices with coefficients from $\mathbb{F}$.

Since matrices represent linear transformation, many standard matrix groups have "*linear*" in their names.

Definition

Thqe general linear group of degree n over R is the set of invertible matrices with coefficients from R:

$$\mathsf{GL}_n(R) = \big\{A \in \mathsf{Mat}_n(R) \mid \det A \neq 0\big\}.$$

The special linear group is the subgroup of matrices with determinant 1:

$$\mathsf{SL}_n(R) = \big\{A \in \mathsf{GL}_n(R) \mid \det A = 1\big\}.$$

An interesting group of order 24

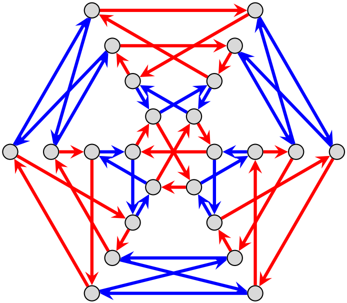
Some interesting finite groups arise as special or general linear groups over $\mathbb{Z}_q$. For example,

$$\mathrm{SL}_2(\mathbb{Z}_3) = \langle A, B \mid A^3 = B^3 = (AB)^2 \rangle = \langle A, B, C \mid A^3 = B^3 = C^2 = CAB \rangle \cong Q_8 \rtimes \mathbb{Z}_3,$$

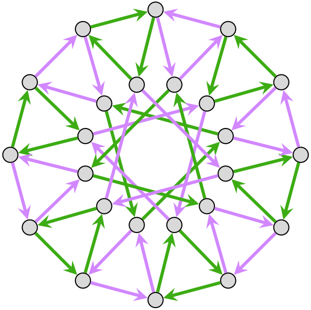
and the matrices A and B can be taken to be

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}.$$

Here's are Cayley graphs for different generating sets:



$\langle R, S \mid R^6 = S^4 = (RS)^3 = I \rangle$



$\langle a, b \mid a^3 = b^3 = (ab)^3 \rangle$

The Hamiltonians

The group $\mathrm{SL}_2(\mathbb{Z}_3)$ can be represented with quaternions. The Hamiltonians are the ring

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}.$$

One way to represent these is with 2×2 matrices over $\mathbb{C}$:

$$\mathbb{H} \cong \left\{ \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix} : z, w \in \mathbb{C} \right\} = \left\{ \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Yet another way involves 4×4 matrices over $\mathbb{R}$:

$$\mathbb{H} \cong \left\{ \begin{bmatrix} a & b & -d & -c \\ -b & a & -c & d \\ d & c & a & b \\ c & -d & -b & a \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Removing 0 from $\mathbb{H}$ defines a multiplicative group $\mathbb{H}^*$ with lots of interesting subgroups.

One of them is the unit quaternions, which physicists assoiciate with points in a 3-sphere:

$$S^3 := \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\}.$$

The group $\mathrm{SL}_2(\mathbb{Z}_3)$ is isomorphic to a subgroup called the binary tetrahedral group,

$$\mathrm{SL}_2(\mathbb{Z}_3) \cong 2\mathrm{T} := \{\pm 1, \pm i, \pm j, \pm k, \tfrac{1}{2}(\pm 1 \pm i \pm j \pm k)\} \leq S^3.$$

Matrix groups over other finite fields

The group $\mathsf{GL}_n(\mathbb{Z}_p)$ consists of the linear maps of the vector space $\mathbb{Z}_p^n$ to itself.

Each one is determined by an ordered basis $v_1, \dots, v_n$ of $\mathbb{Z}_p^n$.

Let's count these. There are:

1. $p^n - 1$ choices for v_1, then
2. $p^n - p$ choices for v_2, then
3. $p^n - p^2$ choices for v_3, and so on...

n. $p^n - p^{n-1}$ choices for v_n.

Therefore,

$$\big|\,\mathsf{GL}_n(\mathbb{Z}_p)\big| = (p^n - 1)(p^n - p)\cdots(p^n - p^{n-1}).$$

These groups have many subgroups, and they often happen to coincide with familiar groups that we have seen.

For example, by "dumb luck",

$$D_9 \cong \left\langle \begin{bmatrix} 16 & 10 \\ 7 & 14 \end{bmatrix}, \begin{bmatrix} 14 & 6 \\ 10 & 3 \end{bmatrix} \right\rangle \leq \mathsf{GL}_2(\mathbb{Z}_{17}), \qquad \mathsf{Dic}_{12} \cong \left\langle \begin{bmatrix} 2 & 7 \\ 7 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 10 \\ 1 & 0 \end{bmatrix} \right\rangle \leq \mathsf{GL}_2(\mathbb{Z}_{11}).$$

Affine groups

Let V be a vector space over a $\mathbb{F}$. A map $L\colon V \to V$ is linear if

$$L(c\mathbf{x} + d\mathbf{y}) = cL\mathbf{x} + dL\mathbf{y}, \qquad \text{for all } x, y \in V \text{ and } c, d \in \mathbb{F}.$$

If $\dim V = n < \infty$, we can write this with an $n \times n$ matrix.

Key point

- A linear map $f\colon V \to V$ has the form $f(\mathbf{x}) = A\mathbf{x}$.
- An affine map $f\colon V \to V$ has the form $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$.

The 1-dimensional general affine group over a field $\mathbb{F}$ as

$$\mathrm{AGL}_1(\mathbb{F}) = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{F},\ a \neq 0 \right\}.$$

The 2-dimensional general affine group can be defined as

$$\mathrm{AGL}_2(\mathbb{F}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ 0 & 0 & 1 \end{bmatrix} : a_{ij}, b_j \in \mathbb{F},\ a_{11}a_{22} - a_{12}a_{21} \neq 0 \right\}.$$

We can encode an affine map of an n-dimensional space V as an $(n+1) \times (n+1)$ matrix:

$$\mathbf{y} = f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}, \qquad \text{as} \qquad \begin{bmatrix} \mathbf{y} \\ 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$$

Other finite groups

The complete classification of finite groups is an impossible task.

However, work along these lines is worthwhile, because much can be learned from studying the structure of groups.

Open-ended question

What group structural properties are possible, what are impossible, and how does this depend on $|G|$?

One approach is to first understand basic "building block groups," and then deduce properties of larger groups from these building blocks, and how to put them together.

In chemistry, "building blocks" are atoms. In number theory, they are prime numbers.

What is a group theoretic analogue of this?

There are several possible answers.

One approach is to study groups that cannot be collapsed by a nontrivial quotient. These are called simple.

The classification of finite simple groups was completed in 2004. It took over 10000 pages of mathematics spread over 500 papers and 50+ years.

p-groups

A different approach to classify groups is to motivated by the following:

> *to understand groups of order $72 = 2^3 \cdot 3^2$, it would be helpful to first understand groups of order $2^3 = 8$ and $3^2 = 9$.*

Definition

If p is prime, then a p-group is any group G of order p^n.

Let's look at small powers of p.

Every group of order p is cyclic, and hence abelian. We can ask:

For what other integers n do there not exist any nonabelian groups?

We don't yet have the tools to answer this. But let's investigate for small powers of p:

Groups of order p^2.

- There are only two: $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \times \mathbb{Z}_p$.

Groups of order p^3. Staring with $p = 2$:

- three are abelian: $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, and $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$
- the dihedral group D_4
- the quaternion group Q_8.

Theorem

For each prime p, there are 5 groups of order p^3.

Surprisingly, the pattern for $p=2$ does not generalize.

Groups of order p^3, for $p>2$

- the Heisenberg group over $\mathbb{Z}_p$,

$$\mathrm{Heis}(\mathbb{Z}_p) := \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z}_p \right\} \cong C_p^2 \rtimes C_p,$$

- another group defined as

$$G_p := \left\{ \begin{bmatrix} 1+pm & b \\ 0 & 1 \end{bmatrix} : m, b \in \mathbb{Z}_{p^2} \right\} \cong C_{p^2} \rtimes C_p.$$

These generalize from p^3 to p^{1+2n}, and are called extraspecial p-groups:

$$M(p) = \langle a, b, c \mid a^p = b^p = c^p = (ab)^2 = (ac)^2 = 1,\ ab = abc \rangle,$$
$$N(p) = \langle a, b, c \mid a^p = b^p = c,\ (ab)^2 = (ac)^2 = 1,\ ab = abc \rangle.$$

Groups of order ≤ 30

order	groups	order	groups	order	groups	order	groups
1	C_1	12 (cont.)	A_4	18 (cont.)	$D_3 \times C_3$	24 (cont.)	$Q_8 \times C_3$
2	C_2	13	C_{13}		$C_3 \rtimes D_3$		$D_3 \times C_4$
3	C_3	14	C_{14}	19	C_{19}		$D_3 \times C_2^2$
4	C_4		D_7	20	C_{20}		$C_3 \rtimes C_8$
	C_2^2	15	C_{15}		$C_{10} \times C_2$		$C_3 \rtimes D_4$
5	C_5	16	C_{16}		D_{10}	25	C_{25}
6	C_6		$C_8 \times C_2$		Dic_{10}		$C_5 \times C_5$
	D_3		C_4^2		$\mathrm{AGL}_1(\mathbb{Z}_5)$	26	C_{26}
7	C_7		$C_4 \times C_2^2$	21	C_{21}		D_{13}
8	C_8		C_2^4		$C_7 \rtimes C_3$	27	C_{27}
	$C_4 \times C_2$		D_8	22	C_{22}		$C_9 \times C_3$
	C_2^3		SD_8		D_{22}		C_3^3
	D_4		SA_8	23	C_{23}		$C_9 \rtimes C_3$
	Q_8		Q_{16}	24	C_{24}		$C_3^2 \rtimes C_3$
9	C_9		$D_4 \times C_2$		$C_{12} \times C_2$	28	C_{28}
	$C_3 \times C_3$		$Q_8 \times C_2$		$C_6 \times C_2^2$		$C_{14} \times C_2$
10	C_{10}		$C_4 \rtimes C_4$		D_{12}		D_{14}
	$C_5 \times C_2$		$C_2^2 \rtimes C_4$		Dic_{12}		Dic_{14}
11	C_{11}		DQ_8		S_4	29	C_{29}
12	C_{12}	17	C_{17}		$\mathrm{SL}_2(\mathbb{Z}_3)$	30	C_{30}
	$C_6 \times C_2$	18	C_{18}		$A_4 \times C_2$		D_{15}
	D_6		$C_6 \times C_3$		$\mathrm{Dic}_{12} \times C_2$		$D_5 \times C_3$
	Dic_6		D_9		$D_4 \times C_3$		$D_3 \times C_5$