Math 4120, Midterm 1. October 6, 2021

1. (16 points) Consider the *dihedral group* $D_4 = \langle r, f \mid r^4 = f^2 = 1,\ rfr = f \rangle$.
   (a) Construct the Cayley diagram and subgroup lattice.
   (b) Determine which subgroups are normal.
   (c) Denote the conjugacy classes of the subgroups by circling them on the lattice.
   (d) Find the normalizers of all non-normal subgroups.

   For full credit, justify your answers to Parts (c–d).

2. (12 points) For each of the following statements, determine whether it is true or false. If it is true, then provide a proof. If it is false, provide a counterexample.
   (a) If $K \trianglelefteq H \trianglelefteq G$, then $K \trianglelefteq G$.
   (b) If $K \leq H \leq G$, and $K \trianglelefteq G$, then $K \trianglelefteq H$.
   (c) If $K \leq H \leq G$ and $K \trianglelefteq G$, then $H \trianglelefteq G$.
   (d) If $[G : H] = p$ and p is prime, then $H \trianglelefteq G$.

3. (10 points) Construct the subgroup lattice of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Label the subgroups by generators, and label each edge with the corresponding index $[H : K]$.

4. (10 points) Short answer. No justification necessary.
   (a) Find the order of the element $(1234)(56)$ in S_6.
   (b) What is the smallest n such that S_n has an element of order 18?
   (c) Give an example of a *minimal* generating set of a group that is not a *minimum* generating set.
   (d) Give an example of a subgroup H of a group G such that $N_G(H) = H$.
   (e) Give an example of a subgroup H of a group G such that $H \lneq N_G(H) \lneq G$.

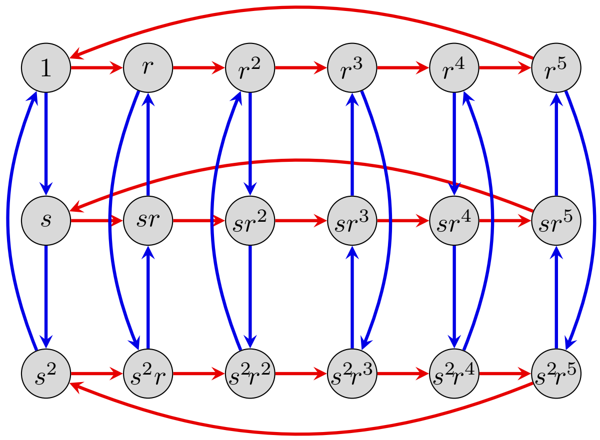
5. (12 points) Let $x \in G$. The *centralizer* of x is the set

   $$C_G(x) = \big\{ g \in G \mid gx = xg \big\}.$$

   Show that $C_G(x)$ is a subgroup of G. Is it necessarily normal? Either prove yes, or provide a counterexample.

6. (10 points) Make a list of all (up to isomorphism) abelian groups of order $360 = 2^3 \cdot 3^2 \cdot 5$, without repetitions. That is, every abelian group of order 360 should be isomorphic to precisely <u>one</u> group on your list.

7. (30 points) Consider the group $G = \langle r, s \rangle$, whose Cayley diagram is shown below. A few extra copies of it are at the bottom of this page, in case you want to use them as scratch paper.



(a) Write a presentation for this group.

(b) Find all left cosets of $H = \langle r \rangle$, and then find all right cosets. Write them as subsets.

(c) Find all left cosets of $K = \langle s \rangle$, and then find all right cosets. Write them as subsets.

(d) Find the normalizers of these groups. Write them by generator(s), and say what familiar group each is isomorphic to.

(e) Find all conjugate subgroups to H and to K. Write each group by generator(s).

(f) What is the center of this group? Justify your answer.

(g) Construct this group as a semidirect product, $A \rtimes_\theta B$. Make sure to define the labeling map $\theta \colon B \to \mathrm{Aut}(A)$, and show any intermediate steps to illustrate your process.