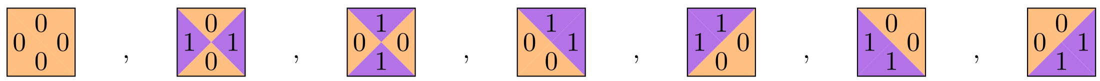
Math 4120, Midterm 2. November 16, 2022

Write your answers for these problems *directly on this paper*. You should be able to fit them in the space given.

1. (14 points) Let S be the following set of 7 "binary squares:"

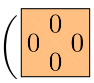
$S = \{$  $\}$

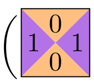
Consider the action of $G = D_4 = \langle r, f \rangle$ on S, where

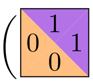
$\phi(r) =$ rotates each square 90° counterclockwise, $\phi(f) =$ reflects each square about a vertical axis.

(a) Draw the *action graph.* (No need to re-draw these squares, just build off of what appears above.)

(b) Find the following:

- stab () =

- stab () =

- stab () =

- $\text{fix}(rf) =$
- $\text{fix}(r^2) =$
- $\text{fix}(r^2 f) =$
- $\text{fix}(r) =$
- $\text{fix}(1) =$
- Average size of $\text{fix}(g)$, where $g \in D_4 =$
- $\text{Fix}(\phi) =$
- $\text{Ker}(\phi) =$

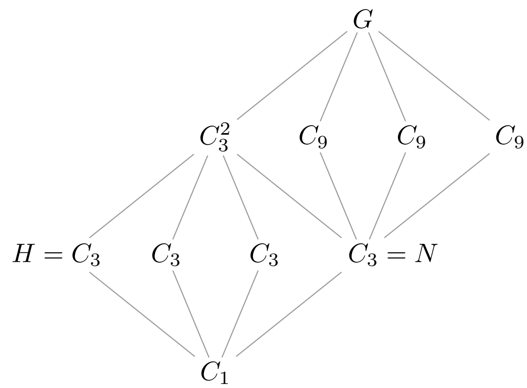
2. (8 points) Suppose the group $C_{15} = \langle r \rangle$ acts on the set S of "binary squares" from the problem above.

(a) Draw all possible distinct action graphs. (In each one, use "•" for the generic elements of S.)

(b) Prove or disprove: this action must have a fixed point.

3. (20 points) Consider the group G whose subgroup lattice is shown below.

Order =



Index =

(a) Find the order and index of each "row" of subgroups, and add it to the diagram above.

(b) What is the quotient G/N isomorphic to, and why?

(c) Which subgroup is the normalizer, $N_G(N)$?

(d) You may assume that H is *not* normal. What is its normalizer, $N_G(H)$, and why?

(e) Partition the subgroups into conjugacy classes G by circling them.

(f) Write G as a direct or semidirect product of its proper subgroups, in as many distinct ways as possible.

(g) Find the commutator subgroup G', and the abelianization, G/G'.

(h) Which subgroup must be $Z(G)$, and why? [*Hint*: What do we know about centers of p-groups? Also, a result from the HW is useful: if $G/Z(G)$ is cyclic, then G is abelian.]

(i) Determine the centralizer $C_G(x)$, where $H = \langle x \rangle$. Justify your answer.

(j) Determine the size of the conjugacy class $\mathrm{cl}_G(x)$, where $H = \langle x \rangle$. Justify your answer.

4. (6 points) Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group and $V_4 = \{e, v, h, vh\}$ be the Klein 4-group. Define a homomorphism

$$\phi\colon Q_8 \longrightarrow V_4, \qquad \phi(i) = v, \quad \phi(j) = h.$$

Find the image of the remaining six elements.

$\phi(1) =$ $\phi(-1) =$ $\phi(k) =$

$\phi(-i) =$ $\phi(-j) =$ $\phi(-k) =$

5. (8 points) Let $\phi\colon G \to H$ be a homomorphism. Show that $N := \mathrm{Ker}(\phi)$ is a subgroup of G, and then show that it is normal.

6. (18 points) The *fundamental homomorphism theorem* (FHT) says that if $\phi\colon G \to H$ is a homomorphism, then $G/\operatorname{Ker}(\phi) \cong \operatorname{Im}(\phi)$.

   (a) Draw a triangular *commutative diagram* that illustrates the FHT.

   (b) Prove the FHT. [*Hint*: Make sure you first define a map $\iota\colon G/N \to H$, where $N = \operatorname{Ker}(\phi)$ by $\iota(gN) = \dots$ (*how?*)]

7. (10 points) Finish the following sentences, so they are *formal* mathematical definitions. Make sure you use terminology like "for all", where appropriate.

   (a) A *homomorphism* ϕ from a group G to H is...

   (b) The *kernel* of a homomorphism ϕ is...

   (c) An *automorphism* ϕ of G is...

   (d) An *action* of a group G on a set S is...

   (e) The *commutator* $[x, y]$ of elements $x, y \in G$ is...

8. (16 points) Fill in the following blanks.

   1. Any homorphism $\phi\colon \mathbb{Z}_8 \to \mathbb{Z}$ must be ______________________________.
   2. A homomorphism $\phi\colon G \to H$ is 1-to-1 iff $\mathrm{Ker}(\phi)$ ______________________________.
   3. If p is prime, then the group $\mathrm{Aut}(\mathbb{Z}_p)$ has order ______________________________.
   4. If G' is the commutator subgroup, then G/G' is the largest ____________________ of G.
   5. For any $n \geq 3$, $D_n \cong A \rtimes B$, a semidirect product of $A =$__________ with $B =$ __________.
   6. The nonabelian group $G =$________ is *not* the semidirect product of any of its proper subgroups.
   7. If $\mathrm{Inn}(G)$ acts on the set of conjugacy classes of G, then the kernel is ____________________.
   8. If G acts on its subgroups by conjugation, $H \in \mathrm{Fix}(\phi)$ if and only if ____________________.
   9. If G acts on itself by by conjugation, $x \in \mathrm{Fix}(\phi)$ if and only if ____________________.
   10. If s and s' are in the same orbit, their stabilizers are ______________________________.
   11. The action of $C_2 \times C_2 \times C_2$ on itself by conjugation has ____________________ orbit(s).
   12. The action of Q_8 on its subgroups by conjugation has ____________________ orbit(s).
   13. If G acts on its subgroups by conjugation, $\mathrm{orb}(H) =$ __________, $\mathrm{stab}(H) =$ __________.
   14. The group A_5 is *simple* because it has exactly ____________________ normal subgroup(s).

9. (Extra credit, 2 points) With the birth of my son Felix exactly four weeks ago today, my family became a group of order 4. What prominent historical mathematician shares his name?