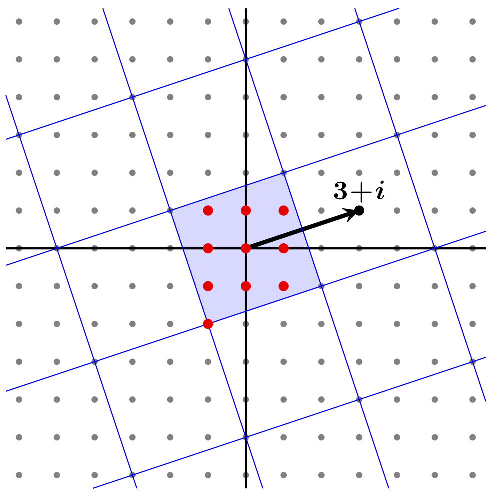
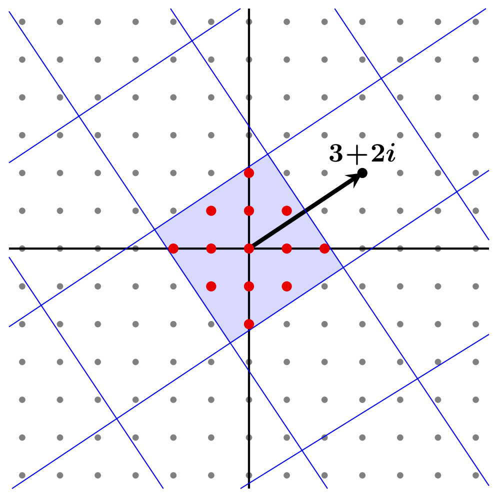
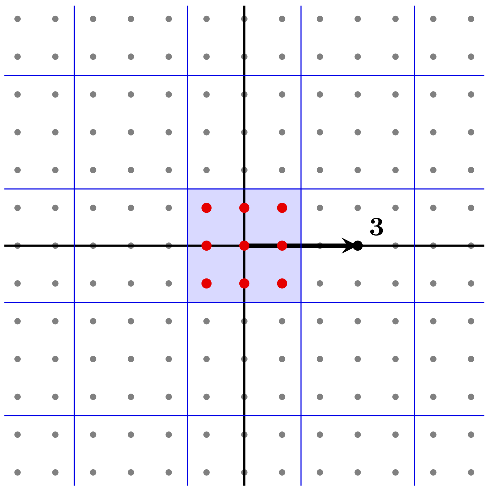
Math 8510, Final Exam. December 15, 2023

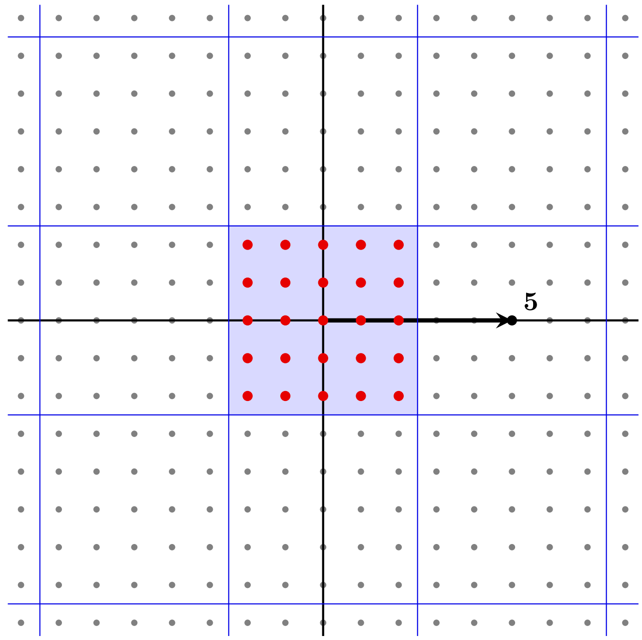
1. (24 points) The following pictures show two quotients of the Gaussian integers $R = \mathbb{Z}[i]$ by an ideal $I = (x)$, for $x = 3 + i$ and $x = 3 + 2i$. The thick red dots denote a full set of coset representatives. For each of these examples, determine whether the ideal (x) is prime, maximal, both, or neither, and what the quotient ring R/I is isomorphic to. Then, determine whether x is irreducible, either by proving that it is, or by demonstrating that it is not by factoring it into irreducibles. Is x prime? Fully justify your answers.





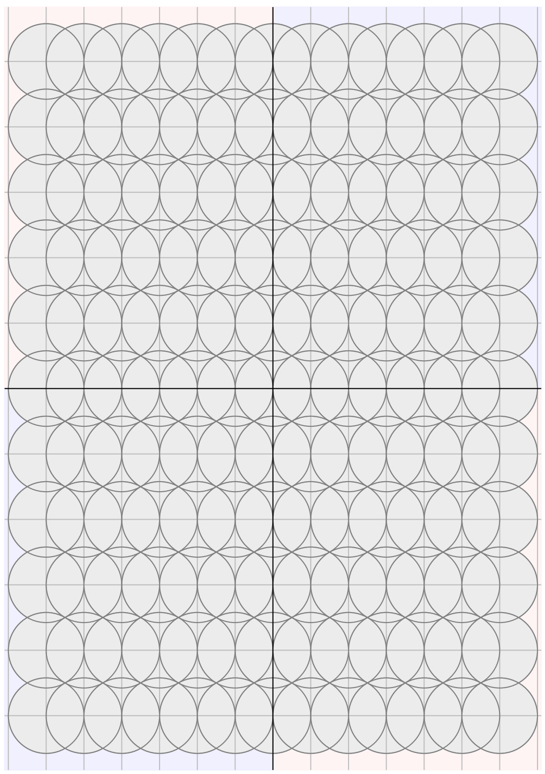
Determine whether the given prime $p \in \mathbb{Z}$ is inert, split, or ramified in $\mathbb{Z}[i]$, and how you know. Then, decide whether the quotient ring $\mathbb{Z}[i]/(p)$ is isomorphic to $\mathbb{Z}_p^2$ or $\mathbb{F}_{p^2}$.

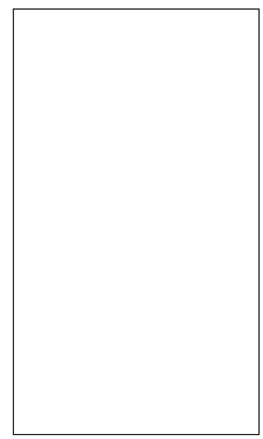




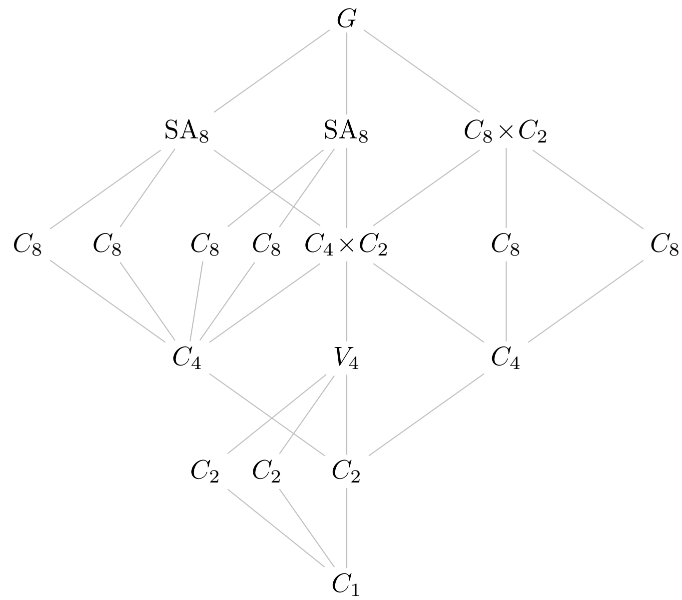
2. (10 points) Two students are arguing over whether the ring $\mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} \mid a, b \in \mathbb{Z}\}$, which is properly contained in the quadratic integer ring R_{-3}, is a PID. One argues that it cannot be, since
$$2 \cdot 2 = 4 = (1 + \sqrt{-3})(1 - \sqrt{-3}).$$
The other demonstrates, using basic trigonometry, how to cover the complex plane with unit balls centered at the elements of $\mathbb{Z}[\sqrt{-3}]$; see below. Therefore, this ring is a Euclidean domain, and thus a PID.

Which student is correct, and why? Point out the flaw in the reasoning of the incorrect student. Fully justify your arguments. A rectangle with height-to-base ratio of $\sqrt{3}$ is provided, in case that is helpful in making your argument.





3. (28 points) Consider the group of order 32 whose subgroup lattice appears below.



(a) $G/(C_4 \times C_2) \cong$ ________________, and $G/C_8 \cong$ ________________ (when it is defined).

(b) The quotients of G by its three order-4 subgroups, reading from left-to-right, are $G/C_4 \cong$ ________________, $G/V_4 \cong$ ________________, and $G/C_4 \cong$ ________________.

(c) The commutator subgroup is $G' =$ _______ and the abelianization is $G/G' \cong$ _______.

(d) The center of G trivially must be contained in the center of all of its subgroups. Recall that the center of SA_8 has order 4, and $\mathrm{SA}_8 / Z(\mathrm{SA}_8) \cong V_4$. Circle this group on the subgroup lattice.

(e) Consider the descending central series $G = L_0 \trianglerighteq L_1 \trianglerighteq L_2 \trianglerighteq \cdots$. Determine which order-4 subgroup L_1 is, with justification, and mark this on the subgroup lattice.

(f) It is now possible to determine the ascending and descending central series, by inspection. Mark these on the subgroup lattice, and provide justification. You may cite basic properties that we proved, such as (i) p-groups are nilpotent, (ii) the ascending and descending central series have the same length, and (iii) $L_k \leq Z_{n-k}$ for all $k \geq 0$.

(g) For each non-normal subgroup H, circle its conjugacy class, $\mathrm{cl}_G(H)$.

(h) What is the inner automorphism group, $\mathrm{Inn}(G)$, isomorphic to, and why?

(i) Is G the semidirect product of any of its nontrivial proper subgroups? Why or why not?

4. (10 points) Let R be a ring with unity. Show that if an ideal I contains a unit, then $I = R$.

5. (16 points) Fill in the following blanks. Assume throughout that G is a nontrivial group.

    1. The smallest nonabelian group is ____.
    2. The equality $aH = bH$ of left cosets implies the equality ________________ of right cosets.
    3. A subgroup $H \le G$ is normal iff its normalizer $N_G(H)$ is ____________________________.
    4. If $xH \neq Hx$ for all $x \notin H$, then $N_G(H)$ (is)(is not)(need not be) the trivial group.
    5. If $\mathrm{cl}_G(H)$ has size 1 for every $H \le G$, then G (is)(is not)(need not be) abelian.
    6. If $\mathrm{cl}_G(g)$ has size 1 for every $g \in G$, then G (is)(is not)(need not be) abelian.
    7. If $\mathrm{cl}_{A_n}(\sigma) \neq \mathrm{cl}_{S_n}(\sigma)$, then the size of $\mathrm{cl}_{A_n}(\sigma)$ is ____________________________________.
    8. In terms of ideals, the condition that $a \mid b$ is equivalent to ____________________________.
    9. An element m in an integral domain is irreducible if (m) is a maximal ________________.
    10. The ring $\mathbb{Z}\big[\frac{1+\sqrt{-19}}{2}\big]$ is __________________________ but not __________________________.
    11. The quadradic integer ring R_m has finitely many units if and only if ___________________.
    12. For $m < 0$, the quadratic integer ring R_m is _____________ iff $m \in \{-1, -2, -3, -7, -11\}$.
    13. Hilbert's basis theorem says that if $1 \in R$, then R is ________________________ iff $R[x]$ is.
    14. In an integral domain R, an ideal I is prime iff the ideal it generates in $R[x]$ is _________.
    15. If R is a UFD and $f(x) \in R[x]$ irreducible, then $f(x)$ (is)(is not)(need not be) irreducible in $F[x]$, where F is its field of fractions.

6. (12 points) Show that there are no simple groups of order $|G| = 30 = 2 \cdot 3 \cdot 5$.

7. (12 points) Let $\phi\colon G \to H$ be a homomorphism. Prove that $\mathrm{Ker}(\phi)$ is a subgroup of G and that it is normal.

8. (16 points) Consider a chain $N \leq H \leq G$ of normal subgroups of G.

   (a) Why must H/N be normal in G/N?

   (b) Prove the fraction (3rd isomorphism) theorem:

$$(G/N)/(H/N) \cong G/H.$$

9. (16 points) Prove the orbit-stabilizer theorem: if G acts on a set S via $\phi\colon G \to \mathrm{Perm}(S)$, then for any $s \in S$, there is a bijection between $\mathrm{orb}(s)$ and right cosets of $H := \mathrm{stab}(s)$.

10. (16 points) Consider a short exact sequence of groups:

$$1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow 1$$

(a) Show that ι is injective and π is surjective.

(b) For each of the following, determine whether it is true or false. If true, then explain why (you may state a result that implies it; you do *not* need to prove it from scratch). If false, provide an explicit counterexample.

(i) If N and Q are solvable, then G is solvable.

(ii) If N and Q are nilpotent, then G is nilpotent.

11. (16 points) Though this problem could be done in the setting of groups or rings, assume we are speaking of groups. Make sure you properly use the correct language, like "for all" and "there exists," when necessary.

    (a) There are two different, but "dual" ways that a map can *factor through* another map. Give a definition of each, and draw a commutative diagram for each.

    (b) Formally state the (co-)universal property of quotient maps. Include a commutative diagram.

    (c) Use the universal property of quotient maps to show that there is no nontrivial homomorphism $\mathbb{Z}_3 \to \mathbb{Z}_4$.

    (d) Use the universal property of quotient maps to show that the map $\mathbb{Z}_8 \to \mathbb{Z}_{12}$, defined by $1 \mapsto 3$, extends to a homomorphism.

12. (16 points) Suppose $G = Q_8$ acts on a set S of size less than 8.

    (a) Draw the subgroup lattice of Q_8; recall that there are four non-trivial proper subgroups: $\langle i \rangle$, $\langle j \rangle$, $\langle k \rangle$, and $\langle -1 \rangle$.

    (b) Show that the stabilizer of every $s \in S$ must contain -1.

    (c) Show that $\mathrm{Ker}(\phi)$ cannot be trivial.

    (d) Show that there is no embedding $Q_8 \hookrightarrow S_7$.

    (e) Show that there is an embedding $Q_8 \hookrightarrow S_8$.

13. (8 points) What was your favorite topic in the class? Specifically, what did you find the most interesting, and why?