Math 8510, Midterm 1. October 18, 2023

1. (10 points) Complete the following statements, using formal mathematical language and/or notation. Correctly use, e.g., "for all" ($\forall$) and "there exists" ($\exists$) where appropriate.

   (a) A *group action* ϕ of G on a set S is (give a formal definition) ...

   (b) If G acts on S, then the *orbit* of the element $s \in S$ is the set:

   $$\text{orb}(s) = \Big\{ \qquad\qquad\qquad\qquad \Big\}.$$

   In particular, it is a subset of (the group G) (the set S) (neither). [$\longleftarrow$ circle one]

   (c) If G acts on S, then the *stabilizer* of an element $s \in S$ is the set:

   $$\text{stab}(s) = \Big\{ \qquad\qquad\qquad\qquad \Big\}.$$

   In particular, it is a subset of (the group G) (the set S) (neither).

   (d) If G acts on S, then the *fixator* of an element $g \in G$ is the set:

   $$\text{fix}(g) = \Big\{ \qquad\qquad\qquad\qquad \Big\}.$$

   In particular, it is a subset of (the group G) (the set S) (neither).

   (e) If G acts on S, then the *fixed points* of the action is the set:

   $$\text{Fix}(\phi) = \Big\{ \qquad\qquad\qquad\qquad \Big\} = \bigcap \qquad\qquad .$$

   In particular, it is a subset of (the group G) (the set S) (neither). Also, write it as an intersection.

   (f) If G acts on S, then the *kernel* of the action is the set:

   $$\text{Ker}(\phi) = \Big\{ \qquad\qquad\qquad\qquad \Big\} = \bigcap \qquad\qquad .$$

   In particular, it is a subset of (the group G) (the set S) (neither). Also, write it as an intersection.

2. (16 points) Fill in the following blanks.

   1. A homomorphism $\phi\colon G \to H$ is one-to-one if and only if its kernel is ________________.
   2. The ________ elements in the symmetric group S_4 fall into ________ conjugacy classes.
   3. The smallest non-cyclic group is ____.
   4. The smallest non-solvable group is ____.
   5. The smallest non-nilpotent group is ____.
   6. The alternating group A_n is simple, except for _________________________________.
   7. An example of a minimal generating set of S_5 of maximal size is ___________________.
   8. The size of a conjugacy class of a subgroup $H \leq G$ is the index of ________________.
   9. The size of a conjugacy class of an element $g \in G$ is the index of ___________________.
   10. An example of a non-simple group that does *not* decompose as a semidirect product of its proper subgroups is ____.
   11. Normality is not transitive, because $H =$ ________ $\trianglelefteq$ ________ $\trianglelefteq$ ________ $= G$, but $H \ntrianglelefteq G$. (Write both proper subgroups in terms of their generator(s), not their isomorphism type.)
   12. If two elements are in the same orbit of an action, their stabilizers are _______________.
   13. A group is nilpotent if and only if it's the direct product of its ____________________.

3. (8 points) Show that there are no simple groups of order $|G| = 42$.

4. (12 points) Let G be a group. Recall that $\phi \in \mathrm{Aut}(G)$ is an *inner automorphism* if it has the form $\phi\colon g \mapsto x^{-1}gx$.

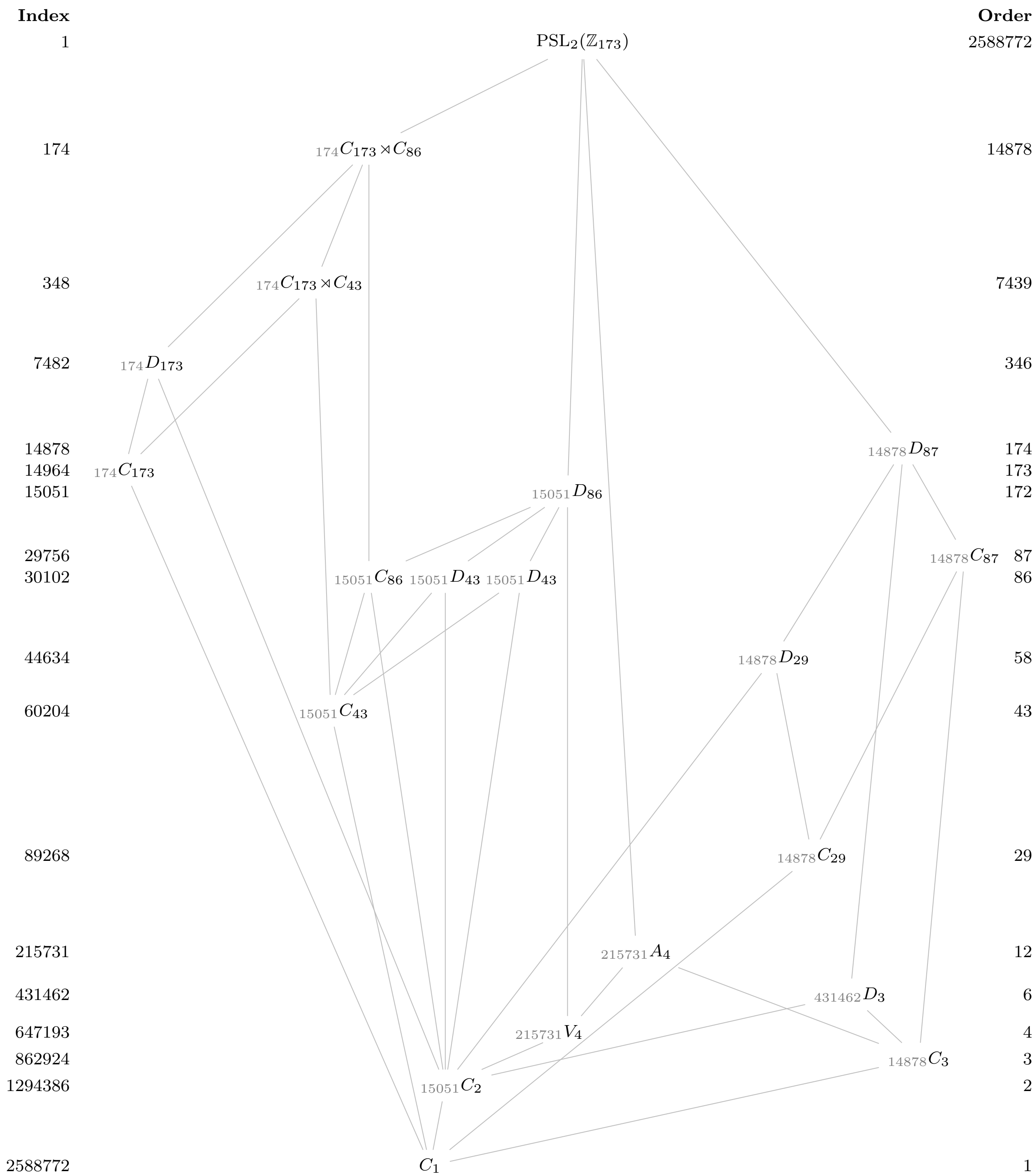
   (a) Show that the set $\mathrm{Inn}(G)$ of inner automorphisms is a subgroup of $\mathrm{Aut}(G)$.

   (b) Show that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.

   (c) Show that $\mathrm{Inn}(G) \cong G/Z(G)$.

5. (24 points) Let G be the *projective special linear group* $\mathrm{PSL}_2(\mathbb{Z}_{173}) \cong \mathrm{SL}_2(\mathbb{Z}_{173})/\langle kI \rangle$ of degree 2 over $\mathbb{Z}_{173}$, whose subgroup lattice appears below. The nodes are conjugacy classes of the subgroups, and the left-subscript denotes their size. This group consists of the 2×2 matrices over $\mathbb{Z}_{173}$ with determinant 1, and two matrices represent the same element if one is a scalar multiple of the other.

   Answer the questions about G that appear on the following page.



(a) The action of $G = \mathrm{PSL}_2(\mathbb{Z}_{173})$ on the set S of its 1,058,074 subgroups by conjugation has ____________________ orbits and ____________________ fixed points.

(b) Of these 1,058,074 subgroups, if we take a randomly selected $g \in G$, what is the expected number of groups H that g normalizes?

(c) Which subgroups of G have the smallest normalizer?

(d) Is G a semidirect product of any of its two proper subgroups? Why or why not?

(e) Let H be one of the subgroups isomorphic to D_{29}, and let G act on its set S of 44,634 cosets. Determine the group of equivariant bijections of this action.

(f) True or false: G has two subgroups $H, K \leq G$ of the same order that are not conjugate.

(g) True or false: G has two isomorphic subgroups $H, K \leq G$ that are not conjugate.

(h) Are all *subgroups* of order 2 conjugate? Why or why not?

(i) Are all *elements* of order 2 conjugate? Why or why not?

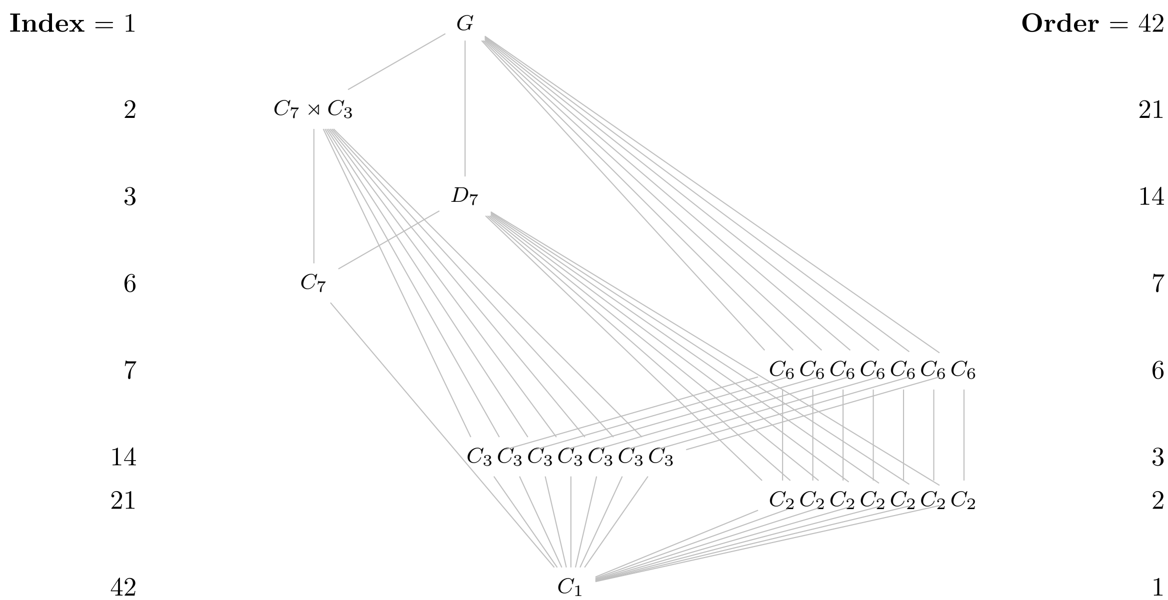
(j) It is elementary to check that $\begin{bmatrix} 7 & 14 \\ 120 & 166 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, which is the identity element in G. How many of the 2,588,772 elements in G commute with $\begin{bmatrix} 7 & 14 \\ 120 & 166 \end{bmatrix}$? Justify your answer.

(k) Is G a simple group?

(l) What is the *inner automorphism* group, $\mathrm{Inn}(G)$, isomorphic to? Justify your answer.

(m) The smallest symmetric group that has a subgroup isomorphic to G is S_{174}. Determine whether $G \leq A_{174}$, and fully justify your answer.

6. (20 points) Answer the following questions about the group whose subgroup lattice is below.



(a) Partition the subgroups into conjugacy classes G by circling them.

(b) Find all N and Q (excluding G and 1) for which G is an extension of Q by N. Write each answer as an exact sequence $1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1$.)

(c) Is G isomorphic to the semidirect product of any of its proper subgroups? If yes, then find all such decompositions. If no, explain why not.

(d) Mark the derived series on the lattice, i.e., write $G^{(0)} =$, $G' =$, $G'' =$,.... Is G solvable?

(e) Mark the descending central series on the lattice, i.e., write $L_0 =$, $L_1 =$, $L_2 =$,.... Is G nilpotent?

(f) In general, it's not possible to identify the center of a group from its subgroup lattice, by inspection. Explain why and how it *can* be determined for this group. Fully justify your answer. [*Hint*: Knowing the descending central series is critical!]

(g) Mark the ascending central series on the lattice, i.e., write $Z_0 =$, $Z_1 =$, $Z_2 =$,....

(h) How many elements does G have of order 2? Are they all conjugate? Justify your answer.

(i) How many elements does G have of order 6? [*Hint*: First, how many elements does C_6 have of order 6?] Are they all conjugate? Justify your answer.

(j) How many elements does G have of order 3? Are they all conjugate? Justify your answer.

7. (10 points) Show that if $|G| = p^n$ for some prime p, then its center is nontrivial, i.e., $|Z(G)| > 1$. [*Hint*: Consider the action of G on...]