Math 8510, Midterm 2. November 29, 2023

1. (12 pts) Write down *complete, formal* mathematical definitions of the following terms.

   (a) A ideal $P \subseteq R$ is *prime* if ...

   (b) A *ring homomorphism* from R to S is ...

   (c) The *co-product* of two rings, R_1 and R_2 is ... [Give the "universal property" definition.]

   (d) The group with *presentation* $G = \langle S \mid R \rangle$, where $S, R \subseteq F_S$ (the free group on S) is ...

   (e) An *initial object* of a category $\mathcal{C}$ is ...

   (f) The *free product* of groups $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ is ...

2. (8 pts) Let R be a commutative ring with 1. By the correspondence theorem, I is a maximal ideal iff R/I is simple. Show that this is equivalent to R/I being a field. [It suffices to show that R is simple iff R is a field.]

3. (8 pts) Show that every nonzero homomorphism $\phi\colon F \to R$ from a field to a ring must be injective.

4. (12 pts) Let $F = \mathbb{F}_{64}$ be a finite field of order $64 = 2^6$.

   (a) Show how to construct F as a quotient ring, R/I. Make sure to explicitly describe what R and I are (give specific generator(s) of I).

   (b) Describe how to add and multiply elements in this field, so that an undergraduate student could do it themselves from your directions alone.

   (c) Which abelian group of order 64 is the additive group F isomorphic to?

   (d) List all abelian groups of order 63 up to isomorphism. Circle the one that is isomorphic to the multiplicative group $F^* = F \setminus \{0\}$.

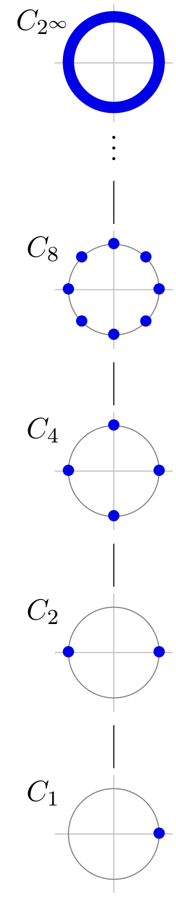
   (e) Which finite fields arise as subfields of F.

5. (10 pts) Let R be a ring with 1, and G a group.

   (a) Carefully finish the statement of Zorn's lemma:

   "*Let* $\mathcal{P} \neq 0$ *be...*

   (b) Use Zorn's lemma to show that every (proper) ideal I of a ring R is contained in a maximal ideal.



   (c) Explain why Zorn's lemma fails to show that every proper subgroup of G is contained in a maximal subgroup. An counterexample is the *Prüfer group*, whose ideal lattice is shown above.

6. (8 pts) Let I be an ideal of a commutative ring R. Recall that the *nilradical* of R is the set of nilpotent elements, or equivalentely, the intersection of nonzero prime ideals:

$$\mathrm{Nil}(R) := \big\{x \in R \mid x^n = 0 \text{ for some } n \in \mathbb{N}\big\} = \bigcap_{0 \neq P \subsetneq R \text{ prime}} P.$$

Use this fact to show that the following two sets of elements are equal; called the *radical* of I, denoted $\sqrt{I}$:

$$\big\{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\big\} = \bigcap_{I \subseteq P \subsetneq R \text{ prime}} P.$$

7. (8 pts) Show that S_3 has presentation $\big\langle a, b \mid a^2 = b^3 = 1,\ aba = b^{-1}\big\rangle$.

8. (9 pts) Let G be a group, $N \trianglelefteq G$, and $\pi\colon G \twoheadrightarrow G/N$ the canonical quotient map.

   (a) Carefully state the *co-universal property of quotient maps*, and include a commutative diagram that illustrates it.

   (b) Formally state and prove the co-universal property of the commutator subgroup $G' = \langle [x,y] : x,y \in G\rangle$ and include a commutative diagram. (Informally, this says that *every homomorphism to an abelian group "factors through"* G/G').

9. (9 pts) Let S be a set, F_1 and F_2 groups, and $\iota_j \colon S \to F_j$ maps (for $j = 1, 2$). Include an appropriate commutative diagram with each part below.

   (a) Carefully define what it means for F_1 to be a free group on S.

   (b) Show that if (F_1, ι_1) and (F_2, ι_2) are both free groups on S, then there is an isomorphism $\phi \colon F_1 \to F_2$ such that $\phi\iota_1 = \iota_2$.

10. (16 points) Fill in the following blanks.

 1. The free group F_S on $S \neq \emptyset$ is abelian iff ________________.
 2. The free product $G_1 * G_2$ is abelian iff ________________.
 3. The free product $G_1 * G_2$ is infinite iff ________________.
 4. An example of a subring that is not an ideal in $\mathbb{Z}[x]$ is ________________.
 5. An example of a subgroup of $\mathbb{Z}[x]$ that is not a subring is ________________.
 6. A commutative ring is an integral domain iff the zero ideal is ________________.
 7. An ideal I is *prime* iff the only zero divisor(s) of R/I is/are ________________.
 8. An ideal I is *primary* iff all zero divisor(s) of R/I is/are ________________.
 9. A example of a primary ideal that is not prime is ________________.
 10. The radical of a primary ideal is ________________.
 11. A example of a non-maximal prime ideal is ________________.
 12. The field of fractions of $\mathbb{F}_p$ has order ________________.
 13. In $R = \mathbb{Z}$, the principal ideal $I = (a) \cap (b)$ is generated by $k =$ ________________.
 14. Every finite integral domain is a(n) ________________.
 15. The smallest $n > 1$ for which there is no field of order n is ________________.
 16. An example of a skew field is ________________.