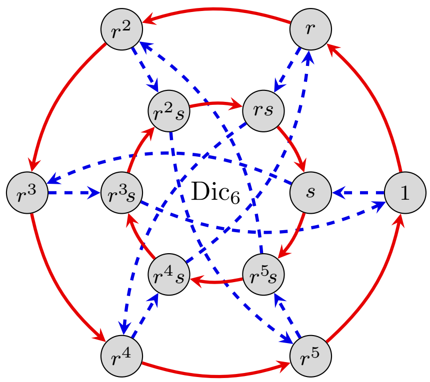
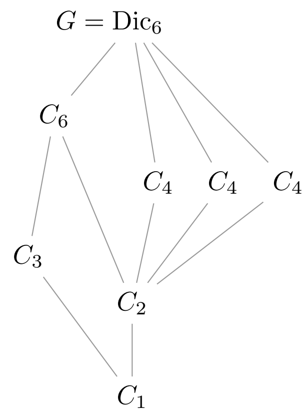
Math 4130, Midterm 2. April 14, 2023

Write your answers for these problems *directly on this paper*. You should be able to fit them in the space given. If not, there is a blank page at the end of this exam.

1. (20 points) A Cayley graph and subgroup lattice of the dicyclic group $G = \mathrm{Dic}_6 = \langle r, s \mid r^6 = s^4 = 1, rsr = s, s^3 = r^2 \rangle$ of order 12 is shown below.





(a) The *center* of G is ____________________. [*Hint*: The Cayley graph is included in case you forget!]

(b) Find all (nontrivial) ways that G can be written as an extension of Q by N. For each one, determine whether it is a central, abelian, or split extension (it can be more than one of these), with justification.

(c) Is G isomorphic to a (nontrivial) direct or semidirect product of its subgroups? Why or why not?

(d) Find all *composition series* of G, and the composition factors of each one.

(e) Find the *derived series* of G, and the *abelianization*, G/G'.

(f) Find the *ascending central series* of G.

(g) Find the *descending central series* of G.

(h) Is G solvable? Nilpotent? Why or why not?

2. (16 points) Finish the following formal mathematical definitions.

   (a) A group G is an *extension of Q by N if* ...

   (b) A group G is *simple* if ...

   (c) A *subnormal series* for G is ...

   (d) A *normal series* for G is ...

   (e) A *compositon series* for a group G is ...

   (f) The *commutator* of two elements x and y is ...

   (g) The *commutator subgroup* of G is ...

   (h) The *derived series* of a group G is...

   (i) An *sequence* of group homomorphisms $\cdots \xrightarrow{\phi_0} G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} G_3 \xrightarrow{\phi_3} \cdots$ is *exact* if...

3. (14 points) Fill in the following blanks.

   1. The commutator subgroup G' is the smallest subgroup such that ________________.
   2. A group G is solvable iff every composition factor is ________________.
   3. Write down the derived series of $G = C_{12}$: ________________.
   4. Write down the derived series of $G = S_5$: ________________.
   5. Write down a composition series of $G = S_5$: ________________.
   6. Write down the ascending central series of $G = D_3$: ________________.
   7. The smallest nonsolvable group is ________________.
   8. The smallest nonnilpotent group is ________________.
   9. An example of a nilpotent group of order *larger* than 8 is ________________.
   10. An example of a group G with derived series $G = G' = G'' = \cdots$ is ________________.
   11. An example of a group G with ascending central series $\langle 1 \rangle = Z_0 \lneq Z_1 = Z_2 = \cdots \neq G$ is ________.
   12. In the ascending central series, Z_{k+1} is the subgroup for which Z_{k+1}/Z_k is the center of ________.
   13. In the descending central series, L_{k+1} is the smallest subgroup s.t. L_k/L_{k+1} is ________ in G/L_{k+1}.
   14. A group is nilpotent iff it's the direct product of its ________________.

4. (10 points) Use the correspondence theorem to prove that if G/N and N are both solvable, then G is solvable. Include a picture that illustrates the proof (almost "without words"), that involves a composition series of both G/N and N. [*Hint*: There are several equivalent condidtions of G being solvable in terms of subnormal series; *make sure you use the right one!*]

5. (8 points) Consider an exact sequence $1 \longrightarrow N \stackrel{\iota}{\longrightarrow} G \stackrel{\pi}{\longrightarrow} Q \longrightarrow 1$ Prove that if N and Q are solvable, then G is solvable. You may use all results about solvable groups and exact sequences provided you state them.

6. (6 points) Show that if $H \leq G$, then $H' \leq G'$.

7. (6 points) Show that if $\phi\colon G \to H$ is a homomrphism, then $\phi([x,y]) = [\phi(x), \phi(y)]$, for all $x, y \in G$.

8. (20 points) This part of the class is all about group *extensions*. We have seen a number of types of extensions: simple, abelian, central, and (left and right) split. Without formally defining these, *what* do they mean (intuitively) and *why* are they important to the theory of groups? How are they encoded by short exact sequences?

   We have also seen a number of types of "subgroup series", like composition series, the derived series, and the ascending and descending series. Compare and contrast these. What do these represent, and how are they related to extensions? What is the significance of the Jördan-Holder theorem? Think of your target audience as a mathematics graduate student who took graduate algebra a few years ago. That is, someone who is mathematically mature, but doesn't remember the details, or maybe didn't ever "get the big picture".