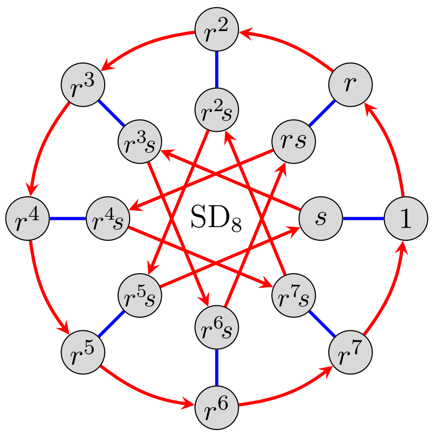
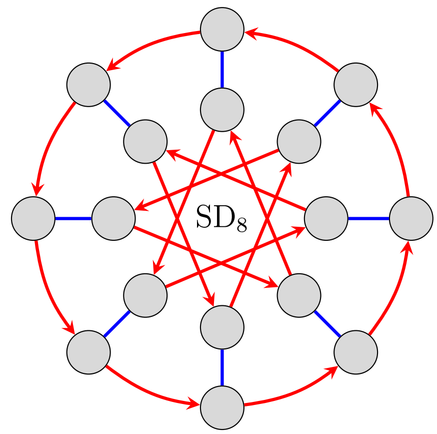
# Math 4120, Final Exam. April 29, 2024

Write your answers for these problems *directly on this paper*. You should be able to fit them in the space given.

1. (22 pts) Answer the following questions about the *semidihedral group* $G = \mathrm{SD}_8$, whose Cayley graph is shown below.




   (a) Write the order of the elements in the nodes in the Cayley graph on the right.

   (b) Write down the left cosets of $H = \langle s \rangle$ in two ways: (i) as $xH$ for some representative $x \in G$, and (ii) as a subset of $G$. Then repeat this for the right cosets.

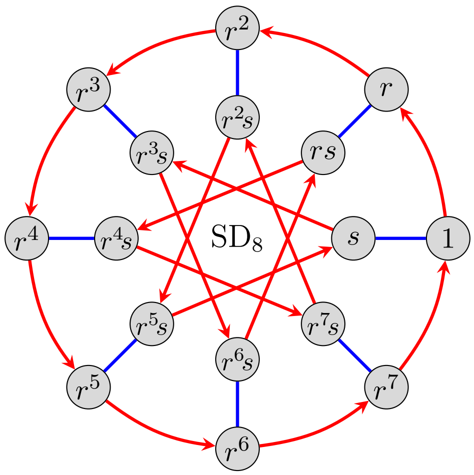
   (c) The *normalizer* of $H$ is $N_G(H) = \langle \qquad\qquad \rangle \cong$__________.

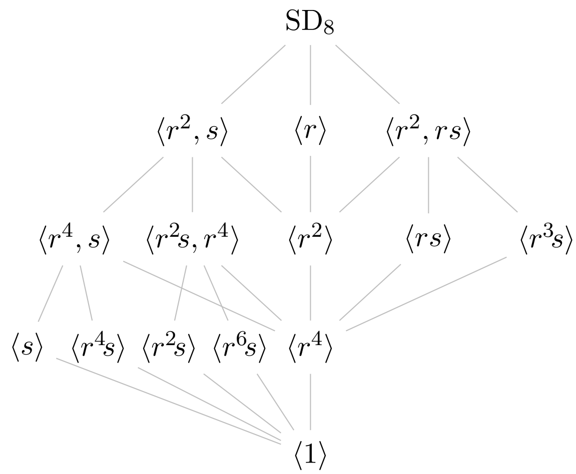
   (d) Find all conjugate subgroups of $H$. Write each subgroup exactly once.

   (e) The subgroup $N = \langle r^2 \rangle$ is normal. Write down its left cosets, and its normalizer.

   (f) Construct a Cayley table, Cayley graph, and subgroup lattice of the quotient $G/N$.

2. (24 pts) Answer the following questions about the same group $G = \mathrm{SD}_8$ from the previous problem, but using its *subgroup lattice*. When asked for a subgroup, write it in terms of generator(s) if $=$ is written, and the isomorphism type if $\cong$ is written.




(a) The subgroup $\langle r^2\rangle$ is isomorphic to __________, and $G/\langle r^2\rangle \cong$__________.

(b) The subgroup $\langle r^4\rangle$ is isomorphic to __________, and $G/\langle r^4\rangle \cong$__________.

(c) $G$ has three order-8 subgroups: $\langle r^2, s\rangle \cong$______, $\langle r\rangle \cong$______, and $\langle r^2, rs\rangle \cong$______.

(d) Find all ways that $G$ can be written as a direct or semidirect product of two of its proper subgroups.

(e) Partition the subgroups into conjugacy classes by circling them on the lattice. Every subgroup should be contained in some circle.

(f) The normalizer of $H = \langle s\rangle$ is $N_G(H) =$________, and $N_G(H)/H \cong$________.

(g) The center of this group is $Z(G) =$____________, and its inner automorphism group is $\mathrm{Inn}(G) \cong$______________.

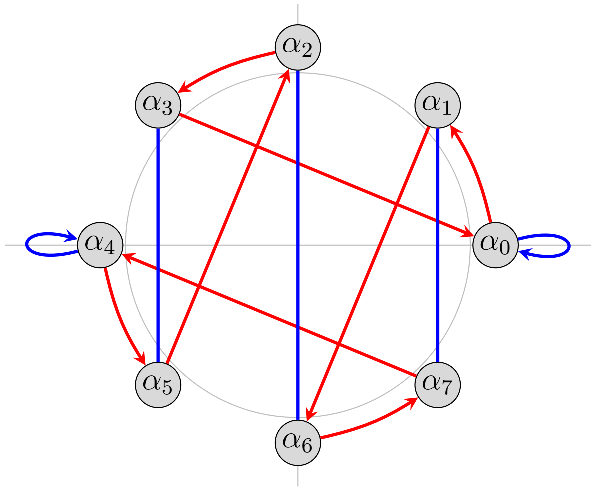
(h) The commutator subgroup is $G' =$______, and the abelianization is $G/G' \cong$______.

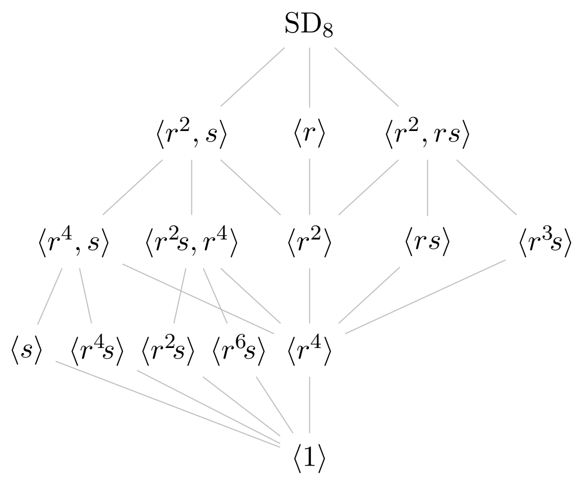
(i) The generator $s \in G$ is conjugate to exactly ________ element(s) of $G$, and thus its centralizer is $C_G(s) =$____________________.

(j) Is $G$ a simple group? Justify your answer in a single sentence. [Do not just give the definition of "simple."]

(k) Exactly ______ of the 15 subgroups of $G$ have normalizer equal to $G$, and ______ of them have normalizer $\langle 1\rangle$.

3. (22 pts) The polynomial $f(x) = x^8 - 2$ has eight distinct roots, $\alpha_0, \alpha_1, \ldots, \alpha_7$, where $\alpha_k = \zeta^k \sqrt[8]{2}$, and $\zeta = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$, a primitive $8^{\text{th}}$ root of unity. An object called the *Galois group* of this polynomial is the familiar semidihedral group $G \cong \mathrm{SD}_8 = \langle r, s \rangle$, from the previous two problems. By a basic result in Galois theory, $G$ acts on the set $S = \{\alpha_0, \ldots, \alpha_7\}$ of roots of $f(x)$. An action graph is shown below, where the position of roots match where they lie on the complex plane.




(a) This action has __________ orbit(s), and (is)(is not) [$\longleftarrow$ *circle one*] transitive.

(b) The orbit containing $\alpha_0$ has size __________, and $\mathrm{stab}(\alpha_0) =$__________.

(c) The orbit containing $\alpha_1$ has size __________, and $\mathrm{stab}(\alpha_1) =$__________.

(d) $\mathrm{stab}(\alpha_2) =$__________ and $\mathrm{stab}(\alpha_3) =$__________.

(e) The automorphism group of $S$, as a $G$-set, is isomorphic to __________.
[*Tip:* At this point, go back and make sure that your answers to the previous parts of this problem agree with what you have for #2(f,g)!]

(f) By inspection, we can compute the *fixators* of the following: $\mathrm{fix}(1) =$ __________,
$\mathrm{fix}(s) =$__________, $\mathrm{fix}(rs) =$ __________, and $\mathrm{fix}(r^2 s) =$ __________.

(g) For this action, $\mathrm{Ker}(\phi) =$__________, and $\mathrm{Fix}(\phi) =$__________.

(h) The subgroup $H = \langle s \rangle$ of $G$ has order __________, and index __________.
It has __________ right coset(s), and __________ left coset(s).

(i) Let $G = \mathrm{SD}_8$ act on the right cosets of $H = \langle s \rangle$ by right multiplication. Draw the action graph. Use colors, or solid vs. dashed lines to distinguish generator edges.

4. (12 pts) Answer the following about each of the well-known groups given, and answer each *(is)(is not)* option by circling the correct response. Recall that a group $G$ is *simple* if its only normal subgroups are $G$ and $\langle e \rangle$.

   (a) The group $A_4$ has order __________, it *(is)(is not)* abelian, and *(is)(is not)* simple.

   (b) The group $C_4$ has order __________, it *(is)(is not)* abelian, and *(is)(is not)* simple.

   (c) The group $D_4$ has order __________, it *(is)(is not)* abelian, and *(is)(is not)* simple.

   (d) The group $S_4$ has order __________, it *(is)(is not)* abelian, and *(is)(is not)* simple.

   (e) The group $V_4$ has order __________, it *(is)(is not)* abelian, and *(is)(is not)* simple.

   (f) The group $Q_8$ has order __________, it *(is)(is not)* abelian, and *(is)(is not)* simple.

5. (10 pts) Consider the following two statements:

   "*$H \leq G$ is the unique subgroup of order $k$*," and "*$H$ is normal in $G$*."

   (a) Prove that one of these statements implies the other.

   (b) Give an explicit counterexample that demonstrates how the converse fails.

   (c) Explain how the Sylow theorems imply that the converse holds for Sylow $p$-subgroups.

6. (15 pts) Let $G$ be a group of order $pq$, where $p$ and $q$ are primes with $p < q$.

   (a) The group $C_n \times C_m$ is isomorphic to the cyclic group $C_{nm}$ iff ___________________.

   (b) Prove that if $G$ (still assuming $|G| = pq$) is not cyclic, then it must be nonabelian.

   (c) Prove that $G$ cannot be a simple group. (Do the abelian and nonabelian cases separately.)

   (d) Prove that if $G$ is nonabelian, then it is a *semidirect product* of cyclic groups. Name any results (e.g., the Sylow theorems, or the characterization of when $G = NH \cong N \rtimes H$) that you use. Assume that the reader knows what they are.

7. (22 pts) In this problem, you will prove the diamond theorem for rings. Throughout, assume that $R$ is a ring, with a subring $S$ and ideal $I$.

   (a) Show that $S \cap I$ is a ideal of $S$. You may assume that it is a subgroup of $R$.

   (b) Prove the diamond theorem for groups: If $G$ is a group, and $A$ normalizes $B$, then $AB/B \cong A/(A \cap B)$. [You may assume that $B \trianglelefteq AB$ and $(A \cap B) \trianglelefteq A$. Start by defining a map $\phi\colon A \to AB/B\ldots$]

   (c) Prove the diamond theorem for rings: $(S + I)/I \cong S/(S \cap I)$. [*Hint*: You may assume the diamond theorem for groups, even if you did not finish Part (b). You should *only* do the additional step needed to establish it for rings.]

8. (18 pts) Answer the following.

   (a) The smallest *nonabelian* group is ________________, the smallest *noncyclic* group is ______________, and the smallest *nonabelian* simple group is ______________.

   (b) If $G$ is a nonabelian simple group, then its commutator subgroup is ____________.

   (c) The roots of the polynomial ________________________ are the $8^{\text{th}}$ *roots of unity*. They form a group, under multiplication, isomorphic to ______________________.

   (d) Subgroups of abelian groups (are)(are not)(are sometimes) [$\longleftarrow$ *circle one*] normal.

   (e) There are exactly two rings of order 7, up to isomorphism. One of of them is the familiar $\mathbb{Z}_7$. The other one can be defined as the set $\{0, a, 2a, 3a, 4a, 5a, 6a\}$, where $ia + ja = (i + j)a \pmod 7$, and multiplication is defined by $ia \cdot ja =$____________.

   (f) If an ideal $I$ of $R$ contains a unit, then $I$ must be ___________________________.

   (g) Though every nonzero element of the Hamiltonians $\mathbb{H} = \big\{a+bi+cj+dk \mid a, b, c, d \in \mathbb{R}\big\}$ is a unit, $\mathbb{H}$ is not a field, because this ring _____________________________.

   (h) The smallest $n \geq 2$ for which there does *not* exist a field of order $n$ is $n =$_______.

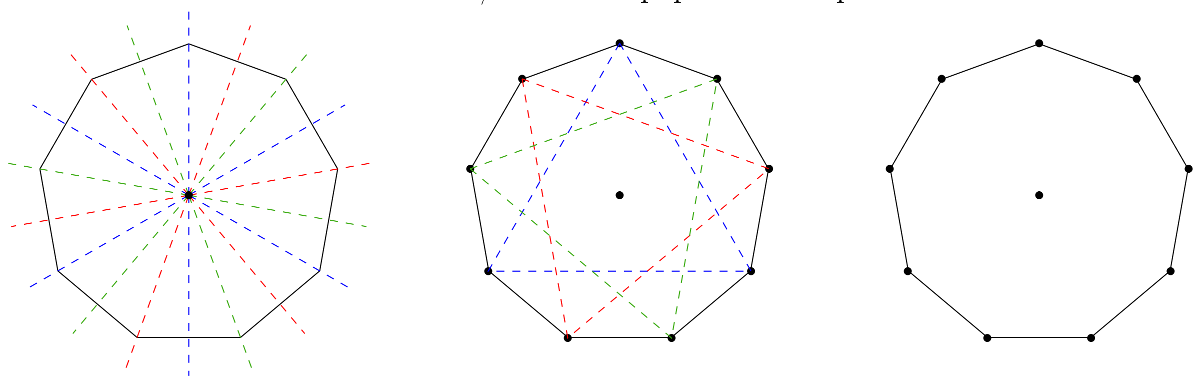
   (i) The units of $\mathbb{Z}_9$ are _________ [$\longleftarrow$ *list them*], and the zero divisors are _________.

   (j) If $I$ is an ideal of $R$, then in the quotient ring $R/I$, the sum of two elements (cosets) is $(x + I) + (y + I) :=$__________, and the product is $(x + I)(y + I) :=$__________.

   (k) By definition, an ideal $I \subseteq R$ is *maximal* if for any other ideal $J$ of $R$, satisfying $I \subseteq J \subseteq R$, either ______________________ or ______________________ must hold.

   (l) By Zorn's lemma, every ideal is contained in a _____________________________.

9. (28 pts) Throughout, let $G = D_9 = \langle r, f \rangle$, the set of symmetries of a regular 9-gon, where $r$ is a counterclockwise $40^\circ$ rotation, and $f$ is a reflection across the vertical (blue) axis, in the picture below (left). Also included below are two other 9-gons, in case you want to use them as a visual reference and/or scratch paper for this problem.



*Tip*: For many of these parts, it is very helpful to think about the group *geometrically*—in terms of the actual symmetries.

(a) The $180^\circ$ rotation *(is)(is not)* [$\longleftarrow$ *circle one*] an element of $D_9$.

(b) The subgroup $\langle f, r^3 f \rangle$, which is generated by two reflections across "blue axes," (see above) is isomorphic to ________________________________.

(c) The subgroup $\langle f, r^2 f \rangle$, generated by two reflections, is isomorphic to __________.

(d) The group $D_9$ has ______ element(s) of order 2, and *(all)(some)(none)* of them are reflections.

(e) The group $D_9$ has __________ element(s) of order 3: __________ [$\longleftarrow$ *list them all*]

(f) The group $D_9$ has ________________ element(s) of order 6: ________________.

(g) The group $D_9$ has ________________ element(s) of order 9: ________________.

(h) $D_9$ has ________________________________ subgroup(s) isomorphic to $V_4$.

(i) $D_9$ has ____ subgroup(s) isomorphic to $D_3$, and ____ subgroup(s) isomorphic to $C_6$.

(j) $D_9$ has ________________________________ subgroup(s) isomorphic to $C_9$.

(k) A *group presentation* for $D_9$ is $\Big\langle r, f \mid \qquad\qquad\qquad\qquad\qquad\qquad \Big\rangle$.

(l) Draw a *cycle graph* for $D_9$. You do not have to label the nodes (but you're welcome to!), and edges should be undirected.

(m) Draw a *Cayley graph* for $D_9 = \langle r, f\rangle$. You don't have to label the nodes with elements, but you may find that doing so will help you later in this problem, or in checking earlier parts.

(n) In a single short sentence, describe what a Cayley graph for $D_9 = \langle s, t\rangle$ would look like, where $s = f$ and $t = rf$ are "adjacent reflections." You do *not* have to actually draw it!

(o) Draw the subgroup lattice for $D_9 = \langle r, f\rangle$ below. Write the subgroups by generator(s), not by isomorphism type. [*Hint*: Organizationally, it is a little easier to write down the groups of order 6 *before* those of order 2.] Circle the conjugacy classes of the non-normal subgroups.

| **Index** = 1 | $D_9 = \langle r, f\rangle$ | **Order** = 18 |
|---|---|---|
| 2 | | 9 |
| 3 | | 6 |
| 6 | | 3 |
| 9 | | 2 |
| 18 | $\langle 1\rangle$ | 1 |

10. (12 pts) Describe each of the following ideals of the given rings in a single sentence, *without* using mathematical notation. For example if $R = \mathbb{Z}[x]$, then the ideal $I = (x)$ is "*the polynomials whose constant term is zero.*"

    (a) In $R = \mathbb{Z}$, the ideal $I = (3)$ is...

    (b) In $R = \mathbb{Z}$, the ideal $I = (4, 6)$ is...

    (c) In $R = \mathbb{Q}$, the ideal $I = (3)$ is...

    (d) In $R = \mathbb{Z}[x]$, the ideal $I = (2)$ is...

    (e) In $R = \mathbb{Z}[x]$, the ideal $I = (x, 2)$ is...

    (f) In $R = \mathbb{Q}[x]$, the ideal $I = (x)$ is...

11. (10 pts) For each of the following rings $R$ and ideals $I$, write down what familiar ring the quotient $R/I$ is isomorphic to. For example, given the ring $R = \mathbb{Z}[x]$ and ideal $I = (x)$, the quotient is $R/I \cong \mathbb{Z}$.

    (a) The quotient of $R = \mathbb{Z}$ by the ideal $I = (3)$ is $R/I \cong$_____________.

    (b) The quotient of $R = \mathbb{Z}$ by the ideal $I = (1)$ is $R/I \cong$_____________.

    (c) The quotient of $R = \mathbb{Z}$ by the ideal $I = (0)$ is $R/I \cong$_____________.

    (d) The quotient of $R = \mathbb{Z}$ by the ideal $I = (3, 7)$ is $R/I \cong$____________.

    (e) The quotient of $R = \mathbb{Z}[x]$ by the ideal $I = (3)$ is $R/I \cong$____________.

    (f) The quotient of $R = \mathbb{Z}[x]$ by the ideal $I = (x)$ is $R/I \cong$____________.

    (g) The quotient of $R = \mathbb{Z}[x]$ by the ideal $I = (3, x)$ is $R/I \cong$___________.

    (h) The quotient of $R = \mathbb{Z}_2[x]$ by the ideal $I = (x^2 + x + 1)$ is $R/I \cong$_______.

    Which of the quotients rings above are *fields* (with $1 \neq 0$, as required). Give you answer by listing a subset of the letters (a),...,(h).

12. (5 points) What was your favorite topic in this class? Specifically, what did you find the most interesting, and why?