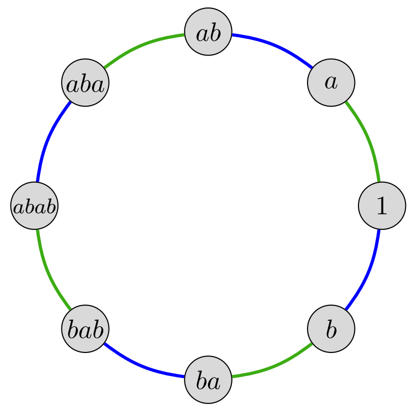
Math 4120, Midterm 1. March 6, 2024

Write your answers for these problems *directly on this paper.* You should be able to fit them in the space given.

1. (10 pts) Let $G = \langle a, b\rangle$ be the group whose Cayley graph is shown below, and let $N = \langle abab\rangle$, which is normal.



   (a) Find all left cosets of N. Write down each one, beside the Cayley graph above, in two ways: *(i)* as xN for some representative $x \in G$, and *(ii)* as a subset of G.

   (b) Construct a Cayley table, Cayley graph, and subgroup lattice of the quotient group G/N. Which familiar group is it isomorphic to?

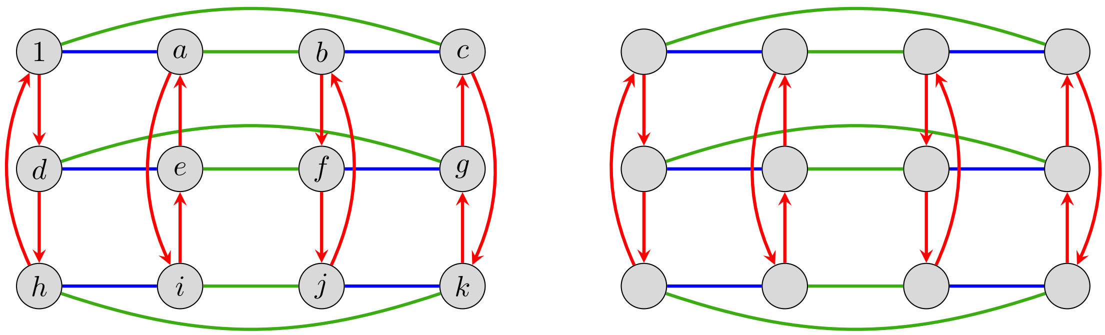
2. (10 pts) Let H and K be subgroups of G, with $K \leq H \leq G$.

   (a) The *index* of H in G, denoted $[G : H]$, is defined as the ________________________.

   (b) Lagrange's thm (an eq'n involving $[G : H]$) says: ______________________________.

   (c) State and prove the *tower law* (an equation relating $[G : K]$, $[G : H]$, and $[H : K]$).

3. (24 pts) A Cayley graph of a mystery group G of order 12 is shown below; let 1 (not e) denote the identity element.



(a) Write the order of each element in the nodes of the blank Cayley graph on the right.

(b) The subgroup $H = \langle a, b\rangle \cong$ ________________, and $K = \langle d\rangle \cong$ ________________.

(c) The subgroup $\langle a, d\rangle$ has order ____________, and is isomorphic to ____________.

(d) The subgroup $\langle b, d\rangle$ has order ____________, and is isomorphic to ____________.

(e) The center of this group is $Z(G) =$ _________. [Write it in terms of generator(s).]

(f) Find all left cosets of $H = \langle a, b\rangle$. Then find all right cosets. Write your answers as subsets of G, or describe them in words (e.g., "the rows" or "the columns").

(g) Find all left cosets of $K = \langle d\rangle$. Then find all right cosets. Write your answers as subsets of G, or describe them in words.

(h) The normalizers are $N_G(H) = \langle \qquad \rangle$ and $N_G(K) = \langle \qquad \rangle$.

(i) Find all conjugate subgroups to H and to K. Write each subgroup only once.

(j) Is H normal? Is K normal?

(k) There are exactly five groups of order 12: C_{12}, $C_6 \times C_2$, D_6, A_4, and Dic_6. Which one is isomorpic to G? You must justify your answer for full credit.

4. (16 pts) Fill in the blanks.

(a) The smallest nontrivial group is ______________________________.

(b) The cyclic group C_{20} has exactly ______________________ subgroups.

(c) A group has exactly two subgroups iff it is ______________________.

(d) The order of an element $g \in G$ is equal to the order of (the subgroup) ______ of G.

(e) The order of the permutation (12)(13) in S_4 is ______________________.

(f) The centralizer of (12) in S_4 is $C_{S_4}((12)) =$ ______________________.

(g) The normalizer of the center $Z(G)$ is ______________________________.

(h) The element $(12)(34) \in S_4$ is conjugate to exactly ________________ permutations.

(i) The dihedral group D_{18} (order 36) has ___ element(s) of order 2 and ___ of order 4.

(j) The dicyclic group Dic_{18} (order 36) has ___ element(s) of order 2 and ___ of order 4.

(k) A subgroup is normal iff its _____________ (a subgroup) is equal to _____________.

(l) The group $\mathbb{Z}$ has exactly ______________________ finite subgroup(s).

(m) $xHx^{-1} = yHy^{-1}$ iff x, y are in the same ______________________________.

5. (8 pts) Prove that every quotient G/H of an abelian group G is abelian.

6. (8 pts) The *multiplicative group modulo* 12 is the set

$$U_{12} = \big\{ k \in \mathbb{Z} \mid 1 \leq k < 12,\ \gcd(12, k) = 1 \big\},$$

where the binary operation is multiplication, modulo 12. Draw a Cayley table and Cayley graph of this group. Then determine which familiar group it is isomorphic to, and justify your answer.

7. (8 pts) Make a list of all abelian groups of order $48 = 2^4 \cdot 3$, up to isomorphism. Every such group should appear exactly once in your list.

8. (10 pts) Let G be a group, N a normal subgroup, and $G/N = \{gN \mid g \in G\}$, the set of left (or equivalently, right) cosets.

   (a) Prove that the binary operation on G/N defined by

   $$aN \cdot bN := abN$$

   is *well-defined*. That is, it does not depend on the choice of coset representative.

   (b) Prove that G/N is a group, using the binary operation defined above.

9. (6 pts) Determine whether the following statement is true or false:

   $$\text{“}\textit{If } H \leq G, \textit{ then its normalizer } N_G(H) \textit{ is normal in } G.\text{”}$$

   If it's true, prove it. If it's false, give an explicit counterexample.