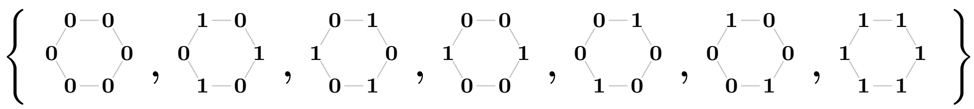
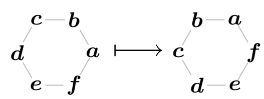
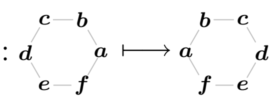
Math 4120, Midterm 2. April 15, 2024

Write your answers for these problems *directly on this paper*. You should be able to fit them in the space given.

1. (20 pts) Consider the following set of seven "binary necklaces", labeled A,...,G for convenience:

$S =$ 

Let $G = D_6 = \langle r, f \mid r^6 = f^2 = (rf)^2 = 1 \rangle$ act on S via the homomorphism

$\phi\colon D_6 \longrightarrow \mathrm{Perm}(S)$, $\phi(r)\colon$  $\phi(f)\colon$ 

(a) Draw the action graph. Write the nodes as A, B, C, D, E, F, G. Use colors, or solid vs. dashed lines to distinguish edges of the generators.

(b) The orbit containing A has size __________, and $\mathrm{stab}(A) =$__________.

(c) The orbit containing B has size __________, and $\mathrm{stab}(B) =$__________.

(d) The orbit containing D has size __________, and $\mathrm{stab}(D) =$__________.

(e) The orbit containing E has size __________, and $\mathrm{stab}(E) =$__________.

(f) $\mathrm{fix}(1) =$ __________, $\mathrm{fix}(r) =$__________, $\mathrm{fix}(r^2) =$ __________,
$\mathrm{fix}(r^3) =$__________, $\mathrm{fix}(f) =$__________, $\mathrm{fix}(rf) =$__________.

(g) For this action, $\mathrm{Ker}(\phi) =$____________, and $\mathrm{Fix}(\phi) =$____________.

2. (9 pts) Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group and S_4 the symmetric group. Define a homomorphism

$$\phi\colon Q_8 \longrightarrow S_4, \qquad \phi(i) = (12), \qquad \phi(j) = (34).$$

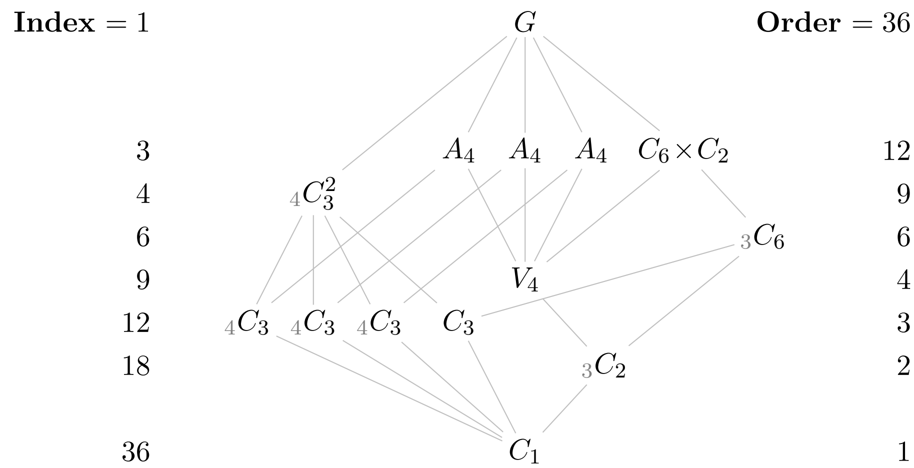
(a) Find the image of the following elements:

$\phi(-1) =$ $\phi(k) =$ $\phi(-i) =$ $\phi(-j) =$ $\phi(-k) =$

(b) $\mathrm{Ker}(\phi) =$______, and $Q_8/\,\mathrm{Ker}(\phi)$ is isomorphic to the familiar group ______.

(c) This homomorphism (is)(is not) [$\longleftarrow$ *circle one*] injective, and (is)(is not) surjective.

3. (26 pts) The subgroup diagram of a group G is shown below.



(a) The group G has ________ subgroups, which fall into ________ conjugacy classes.

(b) The quotient of G by its unique normal subgroup N of order 3 has order ____, and G/N is isomorphic to the familiar group ________________________________.

(c) There are $n_2 =$____________ Sylow 2-subgroup(s), isomorphic to ____________, and $n_3 =$______________ Sylow 3-subgroup(s), isomorphic to ______________.

(d) The commutator subgroup is $G' \cong$_____, and the abelianization is $G/G' \cong$_____.

(e) Find all distinct ways that G can be written as a direct or semidirect product of two of its proper subgroups.

(f) Find the center $Z(G)$, and justify your answer. Though this cannot always be done by inspection, it can in this case, using a result from the previous part.

(g) Let G act on its subgroups by conjugation. This action has ____________ orbit(s) and ________________ fixed point(s).

(h) Let G act on itself by multiplication. This action has ________________ orbit(s) and ________________ fixed point(s).

(i) Let H be a subgroup of order 9, and let G act on the right cosets of H by right multiplication. This action has _________ orbit(s) and _________ fixed point(s).

(j) Still assuming that $|H| = 9$, let G act on the *left* cosets of H by *left* multiplication. This action has ________________ orbit(s) and ________________ fixed point(s).

(k) Let $g \in G$ be an element of order 2. Then g commutes with exactly ___ element(s), and the centralizer of $\langle g \rangle$ is isomorphic to _______________. The centralizer of g is (bigger than)(smaller than)(equal to) [$\longleftarrow$ *circle one*] its normalizer.

4. (15 pts) Let G be a group. For each $a \in G$, define the inner automorphism

$$\phi_a \in \mathrm{Inn}(G) \leq \mathrm{Aut}(G), \qquad \phi_a \colon x \longmapsto a^{-1}xa$$

(a) Show that $\phi_a \phi_b = \phi_{ab}$. [Recall: these are read from *left-to-right*.]

(b) Show that $G/Z(G) \cong \mathrm{Inn}(G)$. [*Hint*: Define a map, show it's a homomorphism...]

5. (10 pts) Prove that any nonabelian group of order $18 = 2 \cdot 3^2$ must be a semidirect product of two proper subgroups. Mention any results (e.g., the Sylow theorems, or the characterization of when $G = NH \cong N \rtimes H$) that you use. Assume that the reader knows what they are.

6. (8 pts) Show that if $\phi\colon G \to H$ is a homomorphism, then $\phi(1_G) = 1_H$. Do not assume that $\phi(g^{-1}) = \phi(g)^{-1}$; that result is proven later.

7. (10 pts) In this problem, you will prove the main part of the correspondence theorem: *every subgroup of G/N has the form H/N, for some subgroup H where $N \leq H \leq G$.*

   (a) First, suppose that $N \leq H \leq G$. Show that $H/N \leq G/N$. [*Hint*: Use the 1-step subgroup test; it's quicker.]

   (b) For the converse, consider that S is a subgroup of G/N. This means that for some *subset* $H \subseteq G$, we have $S = \Big\{ \qquad \mid h \in H \Big\}$. [$\longleftarrow$ *fill in the blank*]

   (c) Show that the subset H in Part (b) is actually a subgroup of G. [*Hint*: See Part (a); the 1-step subgroup test is useful here as well.]

8. (2 pts) In class, we enjoyed the classic hit *Finite Simple Group of Order* __, written and performed by *Klein Four*, an *a capella* group of order ___.